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REPRESENTATION OF THE CITIZENS IN A FEDERAL UNION

Reprezentacja obywateli w unii federalnej

S ł o w a k l u c z o w e: teoria gier, społeczny wybór, reguły głosowania, moc wyborcy.

S t r e s z c z e n i e

Artykuł prezentuje kilka przyczynków, zaproponowanych przez teoretyków teorii gier, ekonomistów, a w szczególności przez ekspertów od społecznych wyborów, dotyczących reguł „dobrego" glosowania w uniach federalnych.

K e y w o r d s: game theory, social choice, voting rules, the power voter.

A b s t r a c t

The article presents several contributions proposed by game theorists and economists, especially social choice experts, on the design of „good" voting rules in federal unions.

Introduction

The objective of this short note is to present several contributions proposed by game theorists and economists, especially social choice experts, on the design of "good" voting rules in federal unions. In federal unions, a decision (or an election) involves often two steps, either because it is impossible to call the electors (decision in the European Union, for example, where a minister represents his country and has a certain number of mandates) or for historical reasons as in the US presidential election case. In both cases, a crucial question is to know how many mandates should be given to countries or states in a two tiers voting system. Very different answers to that question have been adopted by the different federal structures.

The simplest and most natural way is to allocate seats proportionally to the population, in order to give equal rights to each citizen. In the United States, the number of representatives attributed to each state is directly proportional to its population, and is derived after each census[1]. The American Electoral College process follows roughly the same principle: A state i of population n_i gets $A + B$ mandates, where A comes from the number of senators (2 per state) and B comes from its numbers of representatives in the house, which is proportional to its population. In the EU, the number of mandates attributed to each country by the treaty of Nice follows very roughly a $n_i^{1/2}$ law, but with huge fluctuations and a staircase type curve. Each enlargement gave rise to a negotiation among the member states, and no specific rule has never prevailed either on the numbers of mandates per state or about the threshold needed to pass a decision. The European constitution is an attempt to define a more rigorous method, and proposes to conciliate the “one man-one vote” and the “one state-one vote” justifications by the “double key vote”: to be approved, a proposal should be supported by 55% of the countries, their population gathering at least 65% of the EU total population.

Thus, when we look at the American and European cases, we find all kind of systems that try to navigate in between the pure federal system of “one state-one vote” (the second threshold in the European constitution, +2 premium per states in the Electoral college) and the more democratic representation of the states proportionally to their population. Clearly, for all these schemes, the outcome is the result of a political bargain between the small states and the big states. There is also barely no reference to any specific normative criteria that could be used in order to precise what should be the good federal decision process. This apparent confusing situation is also due to the fact that democratic principle (equal treatment for each citizen) is of course easily implemented in a one stage election (for example, imagine a direct popular vote in the US election), but has different solutions in a two-tiers systems. I will now illustrate this point by presenting some of the propositions that have been made by game theorists and economists on this subject.

Measuring Power and Influence

Perhaps, the most widely-publicized normative political claim about two tiers systems comes from game theorists and the voting power literature. Since the

[1] However, there as been of lot of debates in the United States since two centuries in order to find the right mechanism in order to round off the number of representatives per state proportionally to the population. This problem is very well documented, see for example the book by M. Balinski, P. Young, *Fair Representation*, Yale University Press, New Haven 1982. In this paper, by proportionality, we mean that the weight attached to a state is exactly its population, and we will not discuss the rounding off issue.

works of Penrose[2] in 1946 and Banzhaf[3] in 1968, many scholars defend the so called *Penrose square root law*, on the basis that, under a very simple model, the voting power or influence of an individual from state *i* in the Union is proportional to the square root of the population of his home state ($\sqrt{n_i}$); the recent book by Felsenthal and Machover[4] and their papers on the European Union[5] are perfect examples of this tradition.

Let us now describe more precisely the tools and concepts from the power index literature. The simplest model begins with the assumption that *n* voters have to choose between two proposals A and B. These two options are exclusive, abstention is not allowed, and there is no bias in favor of one alternative (such as a statu quo alternative). We next assume that each vote is determined by flipping independently a fair coin randomly; This hypothesis is called the Independence assumption by Straffin[6]. Under this random voting model, all the 2^n vote configurations are equally likely, and the power of voter *j* is simply the proportion of the configurations of the other *n*-1 votes for which voter *j* is decisive. By decisive, we mean that, voter *j*, by changing his vote, can affect the final result[7]. The power of voter *j* is then the number of situations where he is decisive divided by the total number of vote configurations, 2^n. In fact, we have just described a well kown measure of power the (non normalized) *Banzhaf Power index*[8] [3]:

$$\text{Banzhaf power of voter j} = \frac{\text{Number of configurations for which voter j is decisive}}{\text{Total number of voting configurations}}$$

In a federal union, a voter casts his vote in his home state for party *A* or *B*. The winner in state *i* is the party which obtains a majority of votes on his side (abstention is not allowed) among the n_i citizens. Each state *i* is represented at

[2] L.S. Penrose, *The Elementary Statistics of Majority Voting*, "Journal of the Royal Statistical Society" 1946, Vol. 109, Part I, p. 53–57.

[3] J.R. Banzhaf, *Weighted Voting Doesn't Work: a Mathematical Analysis*, "Rutgers Law Review" 1965, no. 13, p. 317–343.

[4] D.S. Felsenthal, M. Machover, *The Measurement of Voting Power*, Edward Elgar Publishing 1998.

[5] D.S. Felsenthal, M. Machover, *The Treaty of Nice and Qualified Majority Voting*, "Social Choice and Welfare" 2001, no. 18, p. 431–464; idem, *Analysis of QM Rules in the Draft Constitution for Europe Proposed by the European Convention 2003*, "Social Choice and Welfare" 2004, no. 23, p. 1–20.

[6] In social choice literature, this assumption is known as the Impartial Culture model. To our knowledge, the term has been coined for the first time by B. Garman and M. Kamien (*The Paradox of Voting: Probability Calculations*, "Behavioral Science 13" 1968, p 306–316) for the analysis of the Condorcet cycle in a three candidate election, though Guilbaud has used the very same assumption as early as 1952 in Social Choice theory.

[7] If the number of mandates is even, ties may occur. A way to avoid such situation is to assume that the number of voters is odd, or to flip a fair coin to take a decision.

[8] J.R. Banzhaf, *Weighted Voting Doesn't Work...*

the federal level by a_i mandates, and the winner in state i catch all these mandates. Then, the position that is officially adopted by the union is the one which obtains a majority of mandates at the federal level. Thus, the probability for a voter to be decisive is the product of the probability that he is decisive in his home state times the probability his state is decisive at the federal level. It is well know[9] that if certain conditions hold – if the number of states is large enough, if no single voter or handful of voters controls almost all the mandates, and there are no discrete features in the weights[10] – that the Banzhaf power of state i with a_i mandates is approximatively proportional to its number of mandates. But in two-tiers voting, we also have to estimate the power of voter j in state i. If n_i is odd, voter j is decisive for

$$\binom{n_i - 1}{\frac{n_i - 1}{2}}$$

configurations among the $2^{n_i - 1}$ possible vote configurations of the other citizens of state i. For n_i large, this can be approximated as $\sqrt{\frac{2}{\pi\,(n_i - 1)}}$. We immediately deduce that:

$$\text{Banzhaf power of voter } j \text{ in state } i \approx a_i \sqrt{\frac{2}{\pi\,(n_i - 1)}} \tag{1}$$

and that equal treatment in term of power is achieved if a_i is proportional to $\sqrt{n_i}$

Clearly, in this model, we do not know nothing about the voters, the states, their socio-economic data, their history. All we know is that, there are some seats, to each seat is attached a certain number of mandates and that there is no bias in favor of a specific motion. We evaluate the distribution of power before the voters enter the room and any discussion can take place. This approach can be defended on the ground that, when designing a voting system or a constitution, we do not have to take into account the present political situation; we may wish the political system to be stable for decades or centuries (in fact a result achieved by the American Constitution). The constitutional planner may wish to propose a voting mechanism under some veil of ignorance, which is exactly what the Independence assumption does.

9 For a detailed explanation, see for example the recent and extremely clear papers by A. Gelman, J.N. Katz and Tuerlinckx, *The Mathematics and Statistics of Voting Power,* "Statistical Sciences" 2002, vol. 17, issue 4, p. 381–467; A. Gelman, J.N. Katz, J. Bafumi, *Standard Voting Power Don't Work: An Empirical Analysis,* "British Journal of Political Sciences" 2004, no. 34, p. 657–674.

10 Feix, Lepelley, Merlin and Rouet (*On the Voting Power of an Alliance and the Subsequent Power of its Member,* "Social Choice and Welfare" 2007, no 2, p. 181–207) have recently shown that these conditions are met for the decision schemes in the European Union.

A second problem with this approach is that, in real life, voter seldom flip coins independently before casting their vote. By analyzing electoral data from the least fifty years, Gelman, Katz and Bafumi[11] have recently showed that the Independence assumption has to be rejected for the elections of the senators, the representatives and president in the United States; similar conclusions are drawn from the electoral data collected over Europe. A way out of this problem is to recognize that the probability of being decisive depends on the probability of the configurations for which a voter is decisive. Then other probability assumptions, also modelling a particular instance of the veil of ignorance hypothesis, can be used. In particular, Straffin[12] and Berg[13] clearly show that if each repartition of the votes between *A* and *B* is equally likely (that is, the probability that A gets 0% of the vote is equal to the probability that he gets 15%, 51% or 89%), the number attached to a configuration with *x* votes for *A* and (*n* – *x*) votes for B is

$$\binom{n}{x}$$

This is Straffin Homogeneity assumption[14]. Then, the right measure of power, that is the probability of being decisive, is now given by the *Shapley-Shubick index*[15]. Next, as there are $\binom{n_i - 1}{(n_i - 1)/2}$ configurations of the other voters which split equally between *A* and *B* in state *i*:

Shapley-Shubick power for player *j* in state $i \approx \dfrac{a_i}{n_i}$

and equal treatment is obtained when the number of mandates is proportional to the population. To conclude, the objective of equalizing the probability of being decisive in two tiers system has no clear answer. The choice of the right apportionment rule is completly driven by the characteristic of the underlying probability model governing the behavior of the voters, which in turn defines a particular measure of the power. Moreover, both models assume that the behavior of the voters is similar across the states.

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[11] A. Gelman, J.N. Katz, J. Bafumi, *Standard Voting Power Don't Work...*

[12] P.D. Straffin, *Homogeneity, Independence and Power Indices*, "Public Choice" 1977, no. 30, p 107-118.

[13] S. Berg, *On Voting Power Indices and a Class of Probability Distributions: With Applications to EU Data*, "Group Decision and Negotiation" 1999, no. 8, p. 17–31.

[14] One should also notice that the Impartial Anonymous Culture used in Social Choice theory to compute the likelihood of the Condorcet paradox gives the same probability when used in binary elections. The term IAC and the model were used first by Gehrlein and Fishburn, *Condorcet's Paradox and Anonymous Preference Profiles*, "Public Choice" 1976, p. 1–18.

[15] L.S. Shapley, M. Shubick, *A Method for Evaluating the Distribution of Power in a Committee System*, "American Political Science Review" 1954, no. 48, p. 787–792.

Efficiency Arguments

The argument that the citizens of the different states should be given equal power, that is equal probability to be decisive, was for a long time the only normative argument proposed by mathematical economists to evaluate the merits of a federal constitution. Recently, several contributions have suggested that some utilitarian notions could play a role too.

Felsenthal and Machover[16] suggest that, for a federal union, the mean difference between the size of the majority camp among all citizens and the number of citizens who agree with the decision taken by the majority of the delegates of the states should be minimized. They show that under the Independence assumption, the square root rule holds again.

Beisbart, Bovens and Hartmann[17] have recently compared seven different possible decision schemes for the European Union on their capacity to select the motions that will have a positive total utility for its citizens, while rejecting the bad ones. They distinguish between the benchmark rules (Simple majority with weights proportional to population, simple majority with equal weights, double majority model, Penrose weights with 62% threshold) and the political rules (Nice treaty, original draft constitution, present draft constitution). For each country, the utility of a policy for the average citizen in state i is drawn from a normal law of mean μ and standard deviation σ. Using computer simulations, Beisbart, Bovens and Hartmann then compute the average utility brought by each decision rule. They also check whether some countries (the small ones, the big ones) are systematically harmed. Again, they discover that no decision rule performs best independently of all model parameters. For bad motions (μ/σ values significatively smaller than 0) the political rules tends to outperform other rules in terms of expected utility, because they are less permissive and thus effectively block motions with too many negative utilities. On the other hand, when $\mu/\sigma > 0$ benchmark rules tend to outperform political rules.

The normative criteria proposed by Barberà and Jackson[18] uses similar ideas, but is more general with respect to several points. They assume that in a two candidate election, the partisans of candidate A get a utility of $u_j = 1$ if it is elected (and 0 otherwise), while the partisans of B get a utility of $u_j = v, v \in [0, +\infty]$ if their preferred candidate is elected (and 0 otherwise). Thus, one camp may en-

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[16] D.S. Felsenthal, M. Machover, *Minimizing the Mean Majority Deficit: The Second Square-root Rule*, "Mathematical Social Sciences" 1999, no. 37, p. 25–37.

[17] C. Beisbart, L. Bovens, S. Hartmann, *A Utilitarian Assessment of Alternative Decision Rules in the Council of Minister*, "European Union Politics" 2005, no. 6, p. 395–419.

[18] S Barbera, M.O. Jackson , *On the Weights of Nations: Assigning Voting Weights in a Heterogeneous Union*, forthcoming in the "Journal of Political Economy" 2006, vol. 114, no 2, p. 317–339.

joy a higher utility when winning. Then, the optimal voting rule for two tiers elections should the one that maximize the efficiency, that is the total expected utility of the voters. Their first results are very general in the sense that they do not depend upon a particular probability model; The *utility efficient* two-tiers voting rule should be such as:

• The quota of the mandates in order to select *A* against *B* is $\frac{v}{v+1}$. This result justifies the existence of a quota superior to one-half when one camp is particularly harmed when defeated. However, interpersonal comparisons of utility govern the level of the quota; no specific rights for a minority is involved in the reasoning.

• The optimal weight of state *i* is proportional to the total expected utility of the voters knowing that *A* is selected:

$$a_i = E\left[\sum_{j=1}^{n_i} uj \,\middle|\, \text{A is elected}\right].$$

Then depending on the assumption that are made to model a priori the behavior of the voters, different efficient weights can emerge. They retrieve that the Independence assumption leads again to the square root rule, while other probability assumption lead to the proportional rule.

Feix, Merlin, Lepelley, Rouet and Vidu[19] propose and study the properties of another normative criteria to evaluate the different apportionment methods in a federal union: An apportionment method is said to be *majority efficient* if it minimizes the probability that a decision is taken with a majority of mandates at the federal level though it is supported by a minority of voters over the whole union. In other words, we wish to minimize the likelihood of the so called *referendum paradox*[20]: A referendum paradox occurs whenever a decision taken by representatives elected in local jurisdictions conflicts with the decision that would have been adopted if the voters had directly given their opinion through a referendum. This criteria seems very natural, as such strange political situations are not only theoretical objects; They often happens, a well known case being the election of George W. Bush against Al. Gore in 2000[21]. Moreover, it is quite obvious that a federal union cannot work if it is plagued too often by these situations: A majority of the citizens would lose confidence in the institutions, leading to a political crisis. It is possible to argue that, as the referendum paradox has been popularized by the media in 2000 and 2004, the majority efficiency

[19] M. Feix, D. Leppeley, V. Merlin, J.L. Rouet, L. Vidu, *Majority Efficient Representation of the Citizens in a Federal Union*, “Mimeo” 2006.

[20] See H. Nurmi, *Voting Paradoxes, and how to deal with them?*, “Springer” 1999.

[21] For other examples in US, United Kingdom and France, see M. Feix, D. Leppeley, V. Merlin, J.L. Rouet, *The Probability of US Presendential Type Election Paradoxes*, “Economic Theory” 2004, no. 23, p. 227–258.

criteria could be more easily accepted by the public opinion than any criteria based upon the utility or the power concepts. The main difference with the utilitarian efficiency criteria presented previously is that Feix et al. do not here take into account the magnitude of the paradox: They try to evaluate the number of the cases for which a majority of voters is frustrated, but do not weight the results by the magnitude of the paradox, either by counting the number of dissatisfied voter (as in Felsenthal and Machover[22]) or adding up the utilities (as in Barbera and Jackson[23] or Beisbart, Bovens and Hartmann[24]).

Though the criteria of majority efficiency is simple to identify and popularize, the search of the apportionment rule which maximizes the majority efficiency is not that simple: The authors have not been able to resolve analytically the problem, and cannot prove mathematically that it points toward a clear optimal apportionment method. They focuss their analysis on a particular class of rules. More precisely, through simulations, they try to identify which parameter δ minimizes the probability of the paradox if we allocate the seats according to the law $a_i = n_i^{\delta}$. Although this formula does not take into account all the possibilities, it covers the pure federal case ($\delta = 0$), the square-root rule($\delta = 1/2$), the proportional case ($\delta = 1$) and even dictatorship ($\delta \to \infty$). Their computer simulations show that under the Independence assumption, the square root rule emerges as the apportionment method that minimizes the likelihood of conflicts between direct and two-tier voting. On the other hand, the Homogeneity assumption points toward $\delta = 1$ and the proportional rule.

Concluding Remarks

The debates about the European Draft Constitution has revived the researches about the choice of the best voting rules in public economy, social choice theory and game theory. The brief review of the literature that I present in this note does not mean to be exhaustive – I did not talk about the representation of minorities[25], the axiomatic approach to two-tiers voting rules[26], etc. I focussed on contribu-

[22] D.S. Felsenthal, M. Machover, *Minimizing the Mean Majority Deficit...*

[23] S Barbera, M.O. Jackson, *On the Weights of Nations...*

[24] C. Beisbart, L. Bovens, S. Hartmann, *A Utilitarian Assessment...*

[25] P. Aghion, A. Alesina, F. Trebbi, *Chossing Electoral Rules: Theory and Evidence from US Cities*, “Mimeo” March 2005; R. Renault, A. Trannoy, *Protecting Minorities through the Average Voting Rule*, “Journal of Public Economic Theory” 2005, no. 7, p. 167.

[26] P.C.Fishburn, *The Theory of Representative Majority Decision*, “Econometrica” 1971, no. 39, p. 273–284; Y. Murakami, *Logic and Social Choice*, Dover, New York 1968; C. Chambers, *Consistent Representative Democracy*, “Games and Economic Behavior” 2008, no. 62, p. 348–363; S. Bervoets, V. Merlin, *Gerrymander – Proof Representative Democracies*, “International Journal of Game Theory” 2012, no. 41, p. 473–488.

tions that arrived at the same conclusion while using different normative criteria: The choice of the best two-tiers voting rule is governed by the underlying assumptions on the behavior of the voters, the Independence assumption always pointing at the square root rule, and the Homogeneity assumption leading to the proportional rule. Also notice that there is no formal argument or proof stating that maximizing the utility, minimizing the conflicts and equalizing the influence would always lead to the same normative recommendation for a given probability assumption.

In my opinion, the moral of this story is that one has to use power indices or other normative assessments about institutions with precaution. Though power indices have been frequently used to measure the influence of the different agents in institutional schemes (the US legislative system, the European legislative systems, the United Nations, etc.) few authors recognize that the probabilistic foundations of these measures have a deep impact on the conclusion, and even fewer tried to motivate the use of one particular assumption compared to other one. When the issue comes to the allocations of mandates among the different members of a political union, I tend to think that the Homogeneity assumption is more realistic and could serve to set some benchmark about what should be a good voting rule. But this result is fragile, and many others arguments on the behavior of the agents could change the picture in different contexts.