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FROM THE COSSERATS MECHANICS BACKGROUNDS TO MODERN FIELD THEORY

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Abstract

In the paper, yet weekly known, Cosserats' original four concepts as follow: the four-time unification of rigid body dynamics, statics of flexible rods, statics of elastic surfaces and 3D deformable body dynamics; the intrinsic formulation based on the local, von Helmholtz symmetry group of monodromy; the invariance under the Euclidean group. The concept of a set of low-dimensional branes immersed into Euclidean space are revalorized and explained in terms of the modern gauge field theory and the extended strings theory. Additionally, some useful mathematical tools that connect the continuum mechanics and the classical field theory (for instance, the convective coordinates, von Mises' "Motorrechnung", the Grassmann extensions, Euclidean invariance, etc.) are involved in the historical explanation that how the ideas were developing themself.

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Interrelations between mechanics backgrounds and old and modern physics

This paper is devoted to explanation of a few yet unknown facts concerning understanding in continuum mechanics the Cosserats idea of “intrinsic description”. In the paper we discuss unknown aspects of three COSSERATS’ papers: *Sur la théorie de l’elasticite* (1896) and *Théorie des corps* déformables (1909), *Note sur la théorie de l’action euclidienne* (1909). Cosserat Brothers, not related with any university, are unknown scientists, even in France. Older one, François Cosserat (1852-1914) was a civil engineer which a professional career has been related with railways building, firstly at the Nord and then the East of France. Eugène Cosserat (1866-1931), younger brother by 14 years, was a professional astronomer with a career spent almost entirely in Toulouse in the south – west of France. From 1896 till the death of François in 1914, the Cosserats published together no less than 21 works in the field of theoretical mechanics. The outstanding scientists like: WILSON (1913), JAUMANN (1918), SCHAEFER (1914), ZHOUNG-HENG and DUBEY (1983), POMMARET (1997), MAUGIN (2014), DELL’ISOLA et al. (2015), NEFF (2019) have made many efforts to approach for full understanding of the Cosserats contribution to the physics of continuum.

In the paper, we develop only yet unknown facts and aspects from the Cosserats. We are basing only on three Cosserats’ papers (COSSERAT, COSSERAT 1896, 1967b, 1909), where these facts were proposed and developed. First, we are turning attention on the pioneering Cosserats’ concept of many-time physical objects (*p*-brane) which are immersed “twice” into a many-dimensional Euclidean space of reference (BADUR 1991). For Cosserats the peripatetic motion means nothing else then difference between two arbitrary immersions – the first immersion is motion-less one and second is governed by a motion described by any group of local symmetry. This approach to modeling of any physical changes, proposed by Cosserats more than a one century ago, in the contemporary physics is dominating and fundamental. Efforts of modern field theory must be simply called as looking for a proper gauge symmetry (CHAICHIAN, NELIPA 1984, POMMARIET 1989, MEISSNER 2013). This fact, that modern gauge theories have their roots within the Cosserats continuum is a little recognizable. Shortly speaking, the mathematical models of mechanical continuum, being the patterns for the string and p-branes theory, are still important and should be replaced in the sector of correct physical theories.

The symmetry group of monodromy motion – a source of intrinsic formulation

The intrinsic formulation of continuum mechanics has many historical sources, one, mostly important, coming from von Helmholtz's old concept concerning a hidden (additional) local symmetry group of motion (HELMHOLTZ 1868, BADUR et al. 2022). This symmetry, frequently called the local (intrinsic) symmetry, usually leads to additional degrees of freedom and additional geometric nonlinearity. The concept of local symmetry itself is a way of introducing into description hidden parameters that cannot appear within the framework of global symmetry and the classical Lagrange or Euler description of continua.

This new possibility for mathematical modeling comes mainly from the fact that the intrinsic formulation needs to postulate some local symmetry which is described within a framework of a continuous Lie group theory. The Lie algebra generated by this intrinsic symmetry should be a base where an observer is located now. This observer cannot measure the classical elements known from Euclidean geometry; therefore, a new type of continuum geometry would be developed. Such a continuum geometry, compatible with the space-time arena, was firstly developed by DARBOUX (1890) and next by CARTAN (1935) (see also: EPSTEIN, DE LEON 1998, HEHL, OBUKHOV 2007, EL NASCHIE 2016). However, only in Cosserats' monograph from 1909, von Helmholtz's concept of intrinsic group of symmetry has finally been stated completely and applied to continua of different dimensions (0D, 1D, 2D, 3D). In very short time, between 1909-1936, owing to efforts such scientists like Poincaré, Appel, Roy, Cartan, Sudria, the method of intrinsic formulation has diffused into whole field theory, especially to electrodynamics and gravitation. Unfortunately, every of these developments run independently, losing a main Cosserats' idea concerning a four-time unification. Therefore, in this report we undertake a problem of revalorization of the common description of zero-, one-, two- and three-dimensional continua.

Since both a rigid body and a thin flexible beam have no-restricted freedom of translation and rotation in space, in 1868, VON HELMHOLTZ (1868), after mathematical works of Klein and Lie, solving the problem of monodromy motion has introduced a local group of symmetry $\mathcal{H}(x,t) = T(3) \lhd SO(3)$. It is a semi-simple multiplication of translation group $T(3)$ with orthogonal group of rotation $SO(3)$. It was first time in the gauge field theory where the local gauging group was introduced to describe intrinsic properties of continuum motion.

Let $\mathbb{M} = \mathbb{M}(\theta_\alpha)$ be an element of group $\mathcal{H}(x,t)$ in a special representation having a form of 4×4 matrixes and six group parameters, written originally as: $\theta_\beta \equiv \{u', v', w', \lambda^x, \lambda^y, \lambda^z\}$. These parameters are physically interpreted by the Cosserats as three displacement components and three components of the (Finger) rotation vector – both referred to the rotating frame reper. Indices: $\beta, \gamma = 1, 2, 3, 4, 5, 6$ go through indices the Lie algebra base. Due to semi-simple multiplication of abelian translation group $T(3)$ and the nonabelian rotation group $SO(3)$ elements of the $\mathcal{H}$ take the form of matrices:

$$\mathbb{M}(\theta_\alpha) = \begin{pmatrix} 1 & u(\theta_\beta) \\ 0 & R(\theta_\gamma) \end{pmatrix}_{4\times 4}, \theta_\beta = \theta_1, \theta_2, \theta_3, \theta_\gamma = \theta_4, \theta_5, \theta_6 \tag{1}$$

The unit element $\mathbb{I} = \mathbb{M}_{|\theta_\alpha=0}$ does not change the fixed frame, but arbitrary element $\mathbb{M} \in \mathcal{H}$ for finite value of θ_β remove the placement by $u(\theta_\beta) = \theta_1, \theta_2, \theta_3$ and rotates the frame by $R(\theta_\gamma)$:

$$R(\theta_\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sin\theta}{\theta} \begin{pmatrix} 0 & -\theta_6 & \theta_5 \\ \theta_6 & 0 & -\theta_4 \\ -\theta_5 & \theta_4 & 0 \end{pmatrix} + \frac{1-\cos\theta}{(\theta)^2} \begin{pmatrix} \theta_4\theta_4 & \theta_4\theta_5 & \theta_4\theta_6 \\ \theta_5\theta_4 & \theta_5\theta_5 & \theta_5\theta_6 \\ \theta_6\theta_4 & \theta_6\theta_5 & \theta_6\theta_6 \end{pmatrix} \tag{2}$$

where the group parameters θ_γ have the interpretation as the Finger rotation vector ($\lambda^x, \lambda^y, \lambda^z$ – in the Cosserats monography; Cosserat, Cosserat 1907) and $\theta^2 = \theta_4^2 + \theta_5^2 + \theta_6^2$ is the length of rotation vector. If we take an element $\mathbb{M}$ near of unit $\mathbb{I}$ (for determine of linear Cosserats elasticity) then $\mathbb{M}$ is a linear function of group parameters θ_β easy visualized within lie algebra matrices $\mathcal{T}_\beta$ as:

$$\mathbb{M}(\theta_\beta) = \mathbb{I} + \begin{pmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ 0 & 0 & \theta_6 & -\theta_5 \\ 0 & -\theta_6 & 0 & \theta_4 \\ 0 & \theta_5 & -\theta_4 & 0 \end{pmatrix} = \mathbb{I} + \theta_\beta \mathcal{T}_\beta, \beta = 1, 2, 3, 4, 5, 6 \tag{3}$$

where six bases of Lie algebra $\mathcal{T}_\beta$ are easily anticipated to be:

$$\mathcal{T}_1 = \begin{pmatrix} 0100 \\ 0000 \\ 0000 \\ 0000 \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} 0010 \\ 0000 \\ 0000 \\ 0000 \end{pmatrix}, \mathcal{T}_3 = \begin{pmatrix} 0001 \\ 0000 \\ 0000 \\ 0000 \end{pmatrix} \tag{4}$$

$$\mathcal{T}_4 = \begin{pmatrix} 00\ 0\ 0 \\ 00\ 0\ 0 \\ 00\ 0\ 1 \\ 00{-}10 \end{pmatrix}, \mathcal{T}_5 = \begin{pmatrix} 000\ 0 \\ 000{-}1 \\ 000\ 0 \\ 010\ 0 \end{pmatrix}, \mathcal{T}_6 = \begin{pmatrix} 0\ 0\ 00 \\ 0\ 0\ 10 \\ 0{-}1\ 00 \\ 0\ 0\ 00 \end{pmatrix} \tag{5}$$

The six bases have intrinsic Lie metrics (so-called the structure constants) related with the commutation relations:

$$[\mathcal{T}_\alpha, \mathcal{T}_\beta] \equiv \mathcal{T}_\alpha \mathcal{T}_\beta - \mathcal{T}_\beta \mathcal{T}_\alpha = C^{\gamma}_{\alpha\beta} \mathcal{T}_\gamma, \alpha, \beta, \gamma = 1, 2, 3, 4, 5, 6$$

where $[\mathcal{T}_\alpha, \mathcal{T}_\beta] = 0$ for $\alpha, \beta = 1, 2, 3$ and $[\mathcal{T}_\alpha, \mathcal{T}_\beta] = [\mathcal{T}_\beta, \mathcal{T}_\alpha] = \in^{\gamma}_{\alpha\beta} \mathcal{T}_\gamma$, for $\alpha, \gamma = 1, 2, 3, \beta = 4, 5, 6$ as well as $[\mathcal{T}_\beta, \mathcal{T}_\gamma] = \in^{\delta}_{\beta\gamma} \mathcal{T}_\delta$ for $\beta, \gamma, \delta = 4, 5, 6$. The structure constants $C^{\gamma}_{\alpha\beta}$ are given by a completely skew-symmetric Ricci object (BADUR 1993a). Next, having group element one and 4-time space-time manifold parametrized by the coordinates ϱ_b one can introduce the covariant-like differentiation within the moving frame (lie algebra intrinsic space) by the Utyama procedure of minimal replacement (KLUGE 1969, HEHL 1973): $\partial_b \to \mathcal{D}_b = \mathbb{I}\partial_b - \mathcal{A}_b$, where $\mathcal{A} = \mathcal{A}_b(\varrho_b) = \mathcal{A}_{\beta b}\, \mathcal{T}_\beta(\varrho_b)$ is a so-called compensating potential which play the similar role to the Christoffel coefficients within a curvilinear coordinates system.

If we take any intrinsic coordinate frame, say $\mathbb{R}'$, then the "covariant derivative" of it must be equal to zero from the definition:

$$\mathcal{D}_b \mathbb{R}' = (\mathbb{I}\partial_b - \mathcal{A}_b)\mathbb{R}' = (\mathbb{I}\partial_b - \mathcal{A}_b)\, \mathbb{M}\mathbb{R} = [(\partial_b \mathbb{M})\mathbb{M}^{-1} - \mathcal{A}_b]\mathbb{R}' = 0 \quad (6)$$

From above it follows that we have the compensating connection – called the "pure gauge potential" in the gauge field theory clearly defined to be:

$$\mathcal{A}_b = (\partial_b \mathbb{M})\mathbb{M}^{-1} = \xi_b \mathcal{T}_1 + \eta_b \mathcal{T}_2 + \zeta_b \mathcal{T}_3 + p_b \mathcal{T}_4 + q_b \mathcal{T}_5 + r_b \mathcal{T}_6 \quad (7)$$

The Cosserats call these objects as measures of "geometric velocities" (BASSET 1895, COSSERAT, COSSERAT1907, 1909a). It is a name borrowed from the dynamics of a rigid body (POISSON 1831, LAME, CLAPEYRON 1833). It best reflects the meaning of the Cosserats concept of "four-time mechanics". In the literature we have found no trace of understanding of this simple fact. In the convected coordinate system ϱ_b, the index $b = t, s, \alpha, i$ means four different cases: rigid body, flexible rods, shells and 3D+time body, therefore the definition (7) is very universal. Notice that from eq. (7) it follows only one correct definition of the Cosserats velocities and strain measures. It, unfortunately, means, for a lot of outstanding authors, that process of looking still new formulations of the Cosserats measures of strain and their energetically coupled stresses should be finished at now.

Also, from definition eq. (7) it follows the integrability conditions, in the fiber bundle geometry commonly written as:

$$(\mathcal{D}_b \mathcal{D}_c - \mathcal{D}_c \mathcal{D}_b)\mathbb{R}' = \mathcal{F}^{\beta}_{bc}\, \mathcal{T}_\beta = \partial_{[b}\partial_{c]}\mathbb{R}' = 0 \quad (8)$$

The new objects $\mathcal{F}_{bc}^{\beta}$ is called „the two-form of curvature" in differential geometry or "the strength field" with the gauge field theory. This condition, reduced only to space, $b = i = x, y, z$, has been discovered by the Cosserats in 1896 (COSSERAT, COSSERAT 1896, §36, eq. A). Notice that $\mathcal{F}_{ij}^{\beta}$, due to screw-symmetry possesses only:

$$\mathcal{F}_{12}^{\beta}, \mathcal{F}_{13}^{\beta}, \mathcal{F}_{23}^{\beta} = 0, \beta = 1, 2, 3, 4, 5, 6 \tag{9}$$

independent components with six values within the Lie algebra. It gives 3×6=18 integrability conditions – the same quality as the Cosserats obtained.

From eq. (8) it follows explicitly that:

$$\mathcal{F}_{bc}^{\beta} \mathcal{T}_{\beta} = \left[-\partial_b \mathcal{A}_c^{\beta} + \partial_c \mathcal{A}_b^{\beta} + C_{\alpha\gamma}^{\beta} \mathcal{A}_b^{\alpha} \mathcal{A}_c^{\gamma}\right] \mathcal{T}_{\beta} = 0 \tag{10}$$

These are time-space integrability conditions (LEHMANN 1964, FERRARESE 1976, BADUR 1993b) which are developed due to the Rankine concept of inertia anisotropy (RANKINE 1851). The *in extenso* form of the Cosserats compatibility conditions has been shown in (BADUR, YANG 1989, STUMPF, BADUR 1990)[1].

Remark 1. Up to Cosserats the continuum mechanics has been developing within the frame of the Cauchy paradigm of the first and second Cauchy laws, where stress tensor was taken to be symmetric and deformation measure was interpreted geometrically as a "change of metric tensor". Since, due to the Cosserats, the first time in the science of deformable material continuum appears the curvature measure p_b, q_b, r_b a reader needs more physical intuition. One, well known, example of intrinsic description of moving frame is the Frenet trihedron (Fig. 1). It is located on a material line in a point M that has three coordinates x, y, z and one intrinsic coordiante s (ϱ_1 in the Cosserats denotations) (FRENET 1847). Any changes in the physical state of this line can by described as the changes with x, y, z coordinates, but in this case the changes with coordinate s are more physically realistic.

A smooth curve $x = x(s) = x_i e_i$ describes the center of a material rod having a finite small cross-section *The fruitful analogy*. Intrinsic coordinate s has the meaning of length of the rod. We can identify all successive derivatives of $x(s)$ with respect to s with vector fields on the curve as: $\delta = \partial_s x = \delta d_1'$ and $\alpha = \partial_s \delta = \partial_{ss} x = \kappa d_2'$. One typically refers to these vector fields as the velocity and acceleration of the curve, even when s does not play the role of time, such as when it represents arc length. Additionally, one calls the curvature of the curve $x(s)$ and a unit vector field d_2' along $x(s)$ that gives its direction,

[1] Anisotropic inertia and its generalization have been analyzed by Prof. S. Forest and co-workers (FOREST, CAILLETAUD, SIEVERT 1997, FOREST, SIEVERT 2003) and BROCATO and CAPRIZ (2001). Also A.C. Eringen and Suhubi has complemented the theory by introducing micro-inertia and renamed it subsequently "micropolar theory" (ERINGEN, SUHUBI 1964).

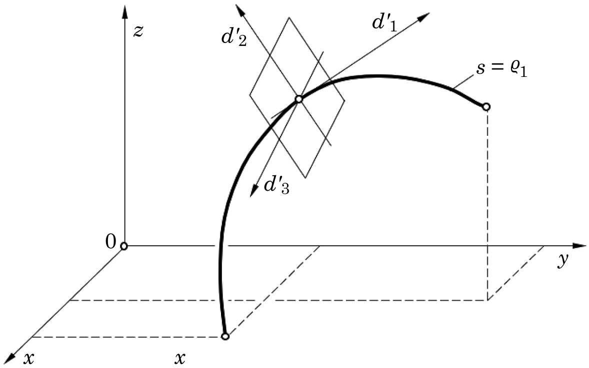

Fig. 1. The Frenet trihedron on the one-dimensional physical line immersed in the Euclidean space
Source: own work.

and which one calls the principal normal vector field for the curve. With s coordinate is related three unite orthogonal, binormal vector $d_3' = d_1' \times d_2'$ that complete the right-handed, orthonormal triad called the Frenet frame field along the curve.

One then finds that the Frenet frame change from point to point that give the equations of the moving frame along a line $x(s)$:

$$\partial_s d_1' = \kappa d_2' - \lambda d_3', \partial_s d_2' = -\kappa d_1' + \tau d_3', \partial_s d_3' = \lambda d_1' - \tau d_2' \tag{11}$$

Here, $\kappa(s)$ is the line curvature, other parameter $\tau(s)$ is referred to as the torsion of the curve. It vanishes if the curve lies in the plane of $d_1'(s)$ and $d_2'(s)$ (viz., the osculating plane to the curve), which will then be the same plane for all s. Parameter $\lambda(s)$ amounts to a rate of rotation about the principal normal $d_2'(s)$, i.e., in the plane of the tangent and binormal. That can be written shortly as:

$$\partial_s d_a' = \mathcal{A}_{ab\,(s)} d_b' = \ \ell_s \times d_a', a, b = 1, 2, 3 \tag{12}$$

where one has two independent notations. The first is a group algebra matrixes notation $\mathcal{A}_{ab\,(s)}$ and second the Poisson-Frenet vector notation $\ell_s = \tau d_1' + \lambda d_2' + \kappa d_3'$. In above:

$$\mathcal{A}_{ab\,(s)} = \begin{pmatrix} 0 & \kappa & -\lambda \\ -\kappa & 0 & \tau \\ \lambda & -\tau & 0 \end{pmatrix} = \tau \mathcal{t}_1 + \lambda \mathcal{t}_2 + \kappa \mathcal{t}_3 \tag{13}$$

Here appears three matrices

$$t_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, t_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, t_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

which are the Lie algebra matrices of $SO(3)$ rotation group. If the motion of orthogonal Frenet frame d_j' are described by the rotation $d_j' = R_{ji}d_i$ of a fixed cartesian frame $d_i, i = x, y, z$ then it is simply to prove that three parameters τ, λ, κ depends on the rotation derivatives as: $\partial_s d_a' = \mathcal{A}_{ab\,(s)} d_b' = \partial_s(R_{ai})d_i = \partial_s(R_{ai})(R_{ib})^{-1}d_b'$. If we consider not only rotation but also displacement of the Frenet frame then, using group description one must exchange group of $SO(3)$ into $T(3) \lhd SO(3)$ and 3×3 matrices algebra (14) into 4×4 algebra (eqs. 4, 5).

Let us note, that the von Helmholtz group action was a subject of von Mises researchers (MISES 1924) who has introduced the Motorrechnung tool for general mechanics in which one resign from 4×4 representation and turns into 6×6 representation of Lie algebra. This motor-calculus can be also conformable for the Cosserats continuum description (SCHAEFER 1967a, 1967b, KESSEL 1970). The equivalence of both: the group description and the Motorrechnung has been shown in (STUMPF, BADUR 1990, BADUR, POVSTIENKO 1998).

How to get continuum of particles?

There is a long tradition in practical using of continuum solid and fluid mechanics. It starts from numerous efforts of Torricelli, Galilee, Boscovich, to exchange a system of N interacting molecules into a concept of deformable continuum (EHLERS et al 2003, PAPENFUSS, FOREST 2006). The first period of doing that was a concept of introducing to the Newton law of motion: $\rho a = f + f_{\text{int}}$ an additional, summary effect of finite molecules spherical interaction in a form of *vis impressa* force:

$$f_{\text{int}} = \textstyle\sum_{n,m} f_{n\to m} = \text{div}(\text{p}) = \text{div}\big(p\delta_{ij}e_i \otimes e_j\big) = \text{grad}\, p \tag{15}$$

where $n, m = 1, 2, 3, \ldots N$ and $i, j = x, y, z$. Finally, owing to Pascal, the notion of continuum pressure has been introduced to continuum approach, where the pressure was imagined as a spherical pressure tensor p who is gluing together body molecules, and a divergence of it represents the sum of forces of mutual interactions[2]. Then, in 1743, Alexis Clairaut by using balance of three forces:

[2] The notion of deformable body has played an important role in the development of the classical field theory (TRUESDELL 1953, TRUESDELL, TOUPIN 1960). Unfortunately, not Boscovich's

pressure (internal stresses) f_{int}, d'Alembert acceleration force f_{cor} and Coriolis centrifugal force f_{cen} written to be: $f_{\text{int}} + f_{\text{cor}} + f_{\text{cen}} = 0$ was able to calculate the flattening of the globe, treated as a spinning drop of liquid magma, became obvious to everyone that the continuum approach must use a stress tensor (LAGRANGE 1762, CAUCHY 1823).

For English researchers, the Pascal prototype was important, so they use the "tensor of pressure" and in all Anglo-Saxon literature it appears with the letter p. It was different on the Continent, what was based on the tradition of Galilee, who studied stretched bodies. Here, finally, Augustine Cauchy in 1823 (CAUCHY 1823), decided to introduce "a tension tensor t" and it's divergence div (t).

The problem of stress tensor symmetry did not exist in the days of Euler and Lagrange. Euler introduced the velocity gradient ($l = \text{grad}\, v$, 1751) and Lagrange introduced the deformation gradient ($F = \text{Grad}\, x$, 1761). Both of these asymmetric objects suggested some kind of stress asymmetry, but both Euler and Lagrange thinking about 3D modelling, have advocated symmetrical measures of deformation – in 1781 Euler introduced a six--component rate deformation tensor: $2d = l + l^T$ and similarly Lagrange in 1787 introduced a symmetric deformation tensor: $2\varepsilon = F + F^T$ (BADUR 2021). It was LAME and CLAPEYRON (1833), and RANKINE (1851) (see history LAMÉ 1852) among the pioneers who developed the invariants in constitutive relations for the symmetric approach.

Yet in XVIII- century Bonnet and de Buat have talked about stress tensor symmetry, but it was not until Navier, when he established the Navier equations for fluids and the Navier equations for solids, that he introduced the angular momentum condition into his Molecular Dynamics. This kind of continuum, both fluid and solid, satisfied angular momentum equally because it used a symmetrical stress tensor. This solution, justified by Navier's Molecular Dynamics, was accepted by Cauchy, Poisson, Lamé, Duhamel, Coriolis, Clapeyron, Poncelet, Liouville, Arago, Barré de Saint-Venant, and others (BADUR 2022).

But in 1852, Ferdinand Reech, the unrecognized French creator of thermodynamics, once again analysed the assumptions of Navier's Molecular

but Fresnel's continuum model of light must be considered as one of the precursors of the nowadays theory of elasticity, on an equal stand with Euler, Lagrange, Du Buat, Navier, Poisson, Cauchy and other French scientist belonging to the period of First French Revolution. Unfortunately, the Frensel concept of continuum physics cannot be accepted any more since it is made under the influence of wrong Newtonian ideas, that only discrete systems of points were still considered. Continuous punctual systems which are simply placed within space-time, appeared in European science with the conception of MacCullagh, which is not sufficient to provide its full power to a model of luminous waves propagation. A similar model of "couples" due to Poinsot have, according to proposition of von Helmholtz, origins in magnetism of polar bodies (BADUR 2021).

Dynamics and came to the conclusion that asymmetrical stress parts and coupled stresses must be involved in the angular momentum balance (REECH 1852). This statement was repeated by Reech's pupil (ANDRADE 1898). New arguments for stress tensor asymmetry and the existence of coupled stresses were presented by PIOLA (1833, 1848), FRENET (1847), MACCULLAGH (1839), THOMSON and TAIT (1883).

Next WALDEMAR VOIGT (1887) studying elastic properties of crystals left open the possibility that there might be internal couple-stresses that acted on the molecules of the lattice. Nowadays we known that Voigt was basically correct in his postulate. For instance, an austenite-martensite phase transformation will induce couple stresses due to appearance of non-symmetric Bain deformations. Such internal couple stresses would then induce an asymmetry in the Cauchy stress tensor. Standing on the experimental data, Pierre Duhem in 1893, postulated the existence of couple stresses (DUHEM 1893). Nevertheless, Duhem having the best knowledge on the Cauchy continuum, was the first who adopted the laws of thermodynamics in the frame of solid and fluid continuum (DUHEM 1901, 1902, 1903,1904, 1905, 1906).

Let us emphasize that the research tools and the idea for the description in the mobile frame, Cosserats took from Lagrange’s and Poisson's mechanics of the material rigid body (LAGRANGE 1762, POISSON 1833). There are different roots then the conventional one in Newtonian mechanics. The Cosserats stood on the shoulders of giants, but none of them were Newton. The concepts of Newtonian mechanics are not taken into account at all – Isaak Newton is systematically removed from the Continent – since his model of gravitation was conceptually wrong. The Cosserats adhere to the paradigm of Lazar Carnot (CARNOT 1793) that the Newtonian mechanics is unfounded and is an accidental fabrication.

Cosserats have accepted, therefore, only the French mathematical tools, borrowed from Lagrange and Poisson mechanics of material point and mechanics of rigid body. They also borrowed the Pascal-Marsenne propositions relative to the notion of “interaction force”, that were applied with the principle of solidification (“rigidification” – due to Reech) of the manifold of material particles. Cosserats have anticipated the possibility of a constitutive relation between the “objective geometrical effort” and the “geometrical moving frame deformation” such the relation was hypothetically first established in a new form of “generalized Hooke’s law” (COSSERAT, COSSERAT 1909a, 1909b) (hyper-elastic material).

The picture of continuum has changed in Cosserats minds – in place of the material point, they called a material trihedron, which is obtained by completing the notion of material point by the addition of three rectangular directions that issue from that point. A continuous medium is then generated by a moving material trihedron. Moreover, it is obvious that the more complex models that are proposed by the molecular theories can give rise to corresponding models for continuous media. It was the line of reasoning a numerous researchers like: KADIĆ and EDELEN (1983), EDELEN and LAGOUDAS (1988), STEINMANN and STEIN (1997), BROCATO and CAPRIZ (2001), CAPRITZ (2010), VARDOULAKIS (2019).

Cosserats solution

The above historical facts are true but cannot be treated as a main motivation for Cosserats work. It is a great mistake that in the literature related to Cosserats is attributed the main goal of establishing Eulerian equations of momentum and angular momentum balance in the Newtonian-like form: $\rho a = f + f_{\text{int}}$ and: $I\epsilon = m + m_{\text{int}}$ where a, ϵ are translational and rotational acceleration. The notions of "momentum" and "moment of momentum" are non-present in the whole Cosserats papers and in the French mechanics. Cosserats develop quite different line of reasoning being supported on the principle of least action. Only in one place: the last chapter of (COSSERAT, COSSERAT 1909a), they are able to obtain from variational of the action the following six Cauchy equation of continuum motion in the Euler description[3]:

$$\rho\dot{v} = \operatorname{div}\sigma + \rho b, I\dot{\omega} = \operatorname{div}\mu + \sigma_{\times} + \rho c \tag{16}$$

These six equations, which are some kind of extensions of Euler's rigid body mechanics onto the continuum case, contain the internal translational interactions of particles: $f_{\text{int}} = \operatorname{div}\sigma$ (see eq. 15), and the rotational interactions of the particles: $m_{\text{int}} = \operatorname{div}\mu + \sigma_{\times}$.

In equation (16), ρ, I are the mass density and the mass inertia tensor, respectively; $\sigma = \sigma_{ij}e_i \otimes e_j, \mu = \mu_{ij}e_i \otimes e_j$ are Cauchy-like, an asymmetrical stress tensor and a couple-stress tensor. Next, $\sigma_{\times} = \epsilon \cdot \sigma$ is an axial vector of $\sigma$and b, c – are specific body force and body couple vectors. If someone asks about existence of c couples then Cosserats response is that the problem of existence of b "force" is much more greater. In the work of the Cosserats, the equations of momentum balance and angular momentum balance are derived via a variational approach at the end of the work (COSSERAT, COSSERAT 1909a, in §73) as a proof that the "intrinsic" approach in Euler's description can be simply connected with the natural description. Notice, that

[3] According to Clifford Truesdell rediscoveries, precisely speaking, one should think about "first and second laws of Euler mechanics" (rigid body science) not about "the first and second Cauchy laws" (deformable continuum science). We agree that the Cosserats approach is totally anti-Newtonian and it is a subtle unification of Euler mechanics laws with the Cauchy mechanics laws (TRUESDELL 1960).

the momentum flux and angular momentum flux tensors σ, μ are originally denoted by the Cosserats with the letters $p = p_{ij} e_i \otimes e_j$ and $q = q_{ij} e_i \otimes e_j$ $(i, j = x, y, z)$ (a natural Euler base indices).

Moreover, the angular momentum equations and angular momentum (16) in the “natural approach” do not appear in the Cosserats monograph (COSSERAT, COSSERAT 1909a) in the main role, they are shown only as the existence of a relationship between the “intrinsic” approach and the “natural” approach, so it cannot be said that they were the main goal of their efforts. If the Cosserats had preferred the “natural Euler description approach”, they would have introduced appropriate Eulerian deformation measures in addition to measures of momentum streams and angular momentum. Equations (16) appeared only in the later works of HELLINGER (1914), JAUMANN (1918) and SIGNORINI (1943) as a generalization of the first and second Cauchy laws.

Remark 2. Let us note, that originally the first and second Cauchy laws (eq. 16) have been written for statics in the natural Euler description approach as: $0 = \operatorname{div} \sigma + \rho b, \sigma_\times = 0$. In the literature there are a few attempts to find analogous equations for the natural Lagrangean description.

Giusseppe Grioli was the first who has rewritten the static version of eq. 16 in the Sudria vector form to be (GRIOLI 1960, eq. 7, 9):

$$\partial_i \varphi_i - f = 0, \partial_i \psi_i + e_i \times \varphi_i + m = 0, \quad i = x, y, z \tag{17}$$

where Eulerian stresses vectors are simply defined as: $\varphi_i = \sigma_{ij} e_j$ and $\psi_i = \mu_{ij} e_j$. Grioli was not able to show any relations between his Eulerian vectors φ_i, ψ_i and the Sudria vectors $\mathcal{E}_b$ and $\mathcal{M}_b$, respectively (in the moving frame). Therefore, he also was not able to find Lagrangean stresses and deformation vectors. Nevertheless, removing any physical argumentation, Grioli, by simple using an analogy with the Cauchy continua ($t = \mathcal{J}^{-1} F S F^{-1}$), has proposed two stresses tensors Φ, Ψ as an analog of the second Piola-Kirchhoff (GRIOLI 1960, eqs SKIBA 1874, LANGE 1885):

$$\sigma_{ij} = \mathcal{J}^{-1} x_{i,r} x_{j,s} \Phi_{rs}, \mu_{ij} = \mathcal{J}^{-1} x_{i,r} x_{j,s} \Psi_{rs} \tag{18}$$

where $\partial_r = \partial / \partial y_r$ are the Lagrangean coordinates, $F = x_{i,r} e_i \otimes e_r$, $\mathcal{J} = |\det F|$. Then having defined $\Phi = \Phi_{rs} e_r \otimes e_s$ and $\Psi = \Psi_{rs} e_r \otimes e_s$ Grioli has proposed the first and second Cauchy laws written within the Lagrangean description as (GRIOLI 1960, eqs KIRCHHOFF 1876, 1877, 1883, DARBOUX 1890):

$$\operatorname{Div} (F\Phi) + f_0 = 0, \operatorname{Div}(F\Psi) + [F\Phi F^T]_\times + m_0 = 0 \tag{19}$$

Axial vector of $[F\Phi F^T]_\times$ by SIGNORINI (1943) and FERRARESE (1959) where written as “one index” notation: $[F\Phi F^T]_\times = [x_{r+1,p} x_{r+2,q} - x_{r+1,q} x_{r+2,p}] \Phi_{pq} e_r$.

From this time, the pull-back transformation (eq. 18) and the operation of taking the axial vector $[F\Phi F^T]_\times$ written firstly by Grioli in the form analogous to elaborated the Cauchy laws (eq. 19) are used within the natural Lagrangean description (TOUPIN 1962, ERINGEN, SUHUBI 1964, KAFADAR, ERINGEN 1971, STEINMANN, STEIN 1997). Also, an analog of the first Piola-Kirchhoff tensor for the couple-stresses was proposed by Grioli (GRIOLI 1960, eq. BASSET 1894) to be: $\Lambda = F\Psi$.

Remark 3. Notice also, that by using some "dimensional" arguments, Grioli has proposed serious modification of the Cosserats energy Lagrangean W in which, originally, only first derivatives of displacement and rotations have appeared. It led to a serious difference with the Cosserats original assumptions which say that if the rotation measures of deformation are equal to zero then also couple stresses must be zero: $\mu_{ij}, \Psi_{rs} = 0$. Grioli has shown that the couple stresses are also the function of second displacement derivative $u_{i,rs}$ (GRIOLI 1960, eq. 57), therefore his reasoning has been changed a thinking about possible form of W. Staring from 1960 the materials with $W = W(u_{i,r};\, u_{i,rs};\, \phi_{i,r})$ are called in the literature the polar continuum (TRUESDELL, TOUPIN 1960, MINDLIN, TRIESTEN 1962, BESSAN 1963, FERRARESE 1971, STEINMANN, STEIN 1997, MAUGIN 1998).

However, for historical precision, due to Clifford Truesdell's efforts, the date of birth of a polar continuum is the year 1686, when Jakob Bernoulli introduced angular momentum as a postulate, independent of balance of momentum (TRUESDELL 1960). It means, that one of the essential features of polar continua is that the stress tensor is not necessarily symmetric, and the balance of angular momentum equation has to be modified accordingly. Owing to Grioli's discovery, we find, that models in which the stress tensor is not symmetric can be regarded as polar-continua. Resuming, in addition of the Cosserats, Grioli asserts, that the non-symmetry of the stress tensor appears also if higher order deformation gradients are included in the specific energy, instead of only the first order gradients. The so-called "material nonlinearities" have nothing to do with the origins on non-symmetry (BESDO 1974).

Remark 4. Note that the writing of the first and second Cauchy laws in the Euler and Lagrange descriptions as well as in the intrinsic approach and simultaneous definitions of stress tensors (Cauchy, first Piola-Kirchhoff, second Piola-Kirchhoff, rotated second Piola-Kirchhoff, etc.) does not mean that we are known which measures of deformation corresponds to them. In such a situation Grioli asks – what are the proper measures of deformation (GRIOLI 1968). Grioli has proposed to consider such measures as:

$$E = \frac{1}{2}(F^T F - 1), \nu = F^T R, \nu_i = F^T \partial_i(R) = F^T R_{,i} \; i = 1, 2, 3 \qquad (20)$$

There are 6+9+27 measures that have no strict relations with the Cosserats measures. Especially, ν_i is not consistent with the Cosserats. But if one defines a tensor $Z_i = \frac{1}{2}R^{-1}R_{,i}$, which is strictly rotational dependent, then the $\nu_i = 2\nu Z_i$. Next having ν and Z_i as the basic measures Grioli has proposed the constitutive relations:

$$\eta = -\frac{\partial W}{\partial \nu}, \tau_i = -\frac{\partial W}{\partial Z_i} \tag{21}$$

According to Grioli, stresses tensors η and τ_i are related with the first Piola-Kirchhoff measures: $F\Phi$ and $\Lambda = F\Psi$ (see, eq. 18). If one defines: $F\Phi = R\eta$ and $\Lambda = F\Psi = -\frac{1}{2}\epsilon_{ijk}R_{rj}\frac{\partial W}{\partial Z_{ik,s}}e_r \otimes e_s$ then the first and second Cauchy laws (eq. 19) can be expressed via responsible strain measures η and τ_i and, finally, a road to complete description within the frame of "natural Lagrangean description" becomes open. This difficult subject was discussed and developed by numerous researchers like: TOUPIN (1962), BESSAN (1963), KOITER (1964), MINDLIN (1964), KAFADAR and ERINGEN (1971), STOJANOVIĆ (1972), STAZI (1976). What is interesting, in the plates and shells theories a way for discovery of Lagrangean measures of deformation is quite different and independent than in 3D. We should follow the concepts of KIRCHHOFF (1850), LANGE (1885), ZERNA (1950), REISSNER and WAN (1968), REISSNER (1972).

Intrinsic approach

The intrinsic description that is using the concept of a mobile reper was the main goal of the Cosserats work. This description has been collected into one unified model mechanics of rigid body (0D), mechanics of deformable strings (1D), mechanics of deformable plates and shells (2D) and, what was great novelty, mechanics of 3D bodes. Unfortunately, this novelty was not developed on the ground of 3D bodies, after 1909, it was further cultivated and gradually became forgotten.

Recall, the intrinsic description involves the introduction of a moving frame of reference where an observer moves and rotates according to the deformation he is supposed to measure. We have two ways of describing the continuum – the first one, the most exploited, is the description called of intrinsic Lagrangean – where the mobile reper is "pushed and rotated" by an element of the local Helmholtz symmetry group: $\mathbb{M} \in \mathcal{H}(x,t) = T(3) \lhd SO(3)$ and the one, little--known "intrinsic Eulerian" where the mobile reper is "pushed and rotated" by means of the inverse element $\mathbb{M}^{-1}$. Both of these descriptions are used by the Cosserats, although the mobile reper description from the Eulerian view is limited by the Cosserats to 3D bodies only.

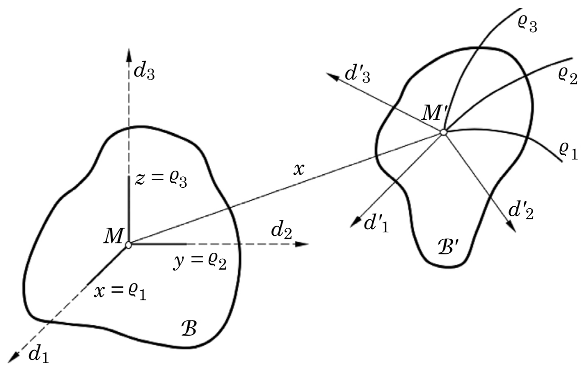

Fig. 2. Cosserats' concept of a moving frame (1896)
Source: own work.

The Lagrangean intrinsic approach, hereinafter referred to as "intrinsic", uses two repers; the first one is moving reper $\mathbb{R}' = \{x', d'_x, d'_y, d'_z\}^T$ to which we refer the description of the body deformation in relation to the second, output, fixed, reper $\mathbb{R} = \{x_0, d_x, d_y, d_z\}^T$:

$$\mathbb{R}' = \begin{Bmatrix} x' \\ d'_1 \\ d'_2 \\ d'_3 \end{Bmatrix} = \begin{pmatrix} 1 & u \\ 0 & R \end{pmatrix} \begin{Bmatrix} x_0 \\ d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \mathbb{M}\mathbb{R}\ (22) \tag{22}$$

The elements of the Helmholtz group u and R are the basic unknowns of the Cosserat model. Physically defining them, these are three components of displacement and three components of rotation. The physical space in which the body is to be inserted has four lagrangean coordinates: ϱ_b where $b = t, x, y, z$ – they are, in a special case, glued to the initial reper $\mathbb{R}$.

The fundamental object was introduced by the Cosserats "ex cathedra" without explaining the physical basis, is described to be: "geometric velocities" (COSSERAT, COSSERAT 1909a, §43). They are formed by differentiating the element of the group according to four coordinates ϱ_b. The Cosserats mark them as: ξ_b, η_b, ζ_b and p_b, q_b, r_b, respectively, with denotation ξ is referring to the first element of the group, η to the second, ζ to the third, p to the fourth, q to the fifth, r to the sixth. Thus, it can be shown that these "geometric velocities" of Cosserats are measures of intrinsic deformation and are collected within a four-vector with six values in the Lie algebra of the group $\mathcal{H}(x,t) = T(3) \lhd SO(3)$. This vector with six values within Lie algebra, according to the tradition of gauge field theory, will be denoted by the letter $\mathcal{A}$.

In the gauge field theory it is called the gauge potential, but in the fiber manifold differential geometry it is called "compensating potential" or "one-form of connection" and, finally, within the moving frame description it is called in French literature “a torsor”, the Germans used the term “dyname”, and the English called it a “wrench”, while the elements of SO(3) were then “screws”. Ericksen and Truesdell re-called this object as “the wryness” (ERICKSEN, TRUESDELL 1958)[4].

If the Lie algebra of $\mathcal{H}(x,t) = T(3) \lhd SO(3)$ consist six matrices $\mathcal{T}_\beta$ ($\beta = 1,2,3$ (translational), $\beta = 4,5,6$ (rotational)) then, the whole Cosserats measures of velocities and deformations can be collected together and simply defined to be:

$$\mathcal{A} = \mathcal{A}_{b\beta}\mathcal{T}_\beta(\varrho_b) = \mathcal{A}_b(\varrho_b) = (\partial_b\mathbb{M})\mathbb{M}^{-1}(\varrho_b) \tag{23}$$

It means that one-form of connection (the gauge potential) $\mathcal{A}$ has four component is space-time and six values in the lie algebra of Helmholtz monodromy group. When we take the time coordinate of 4-time continuum $b = t$ (time) then from eq. (4) one obtains the Cosserats kinematic velocities (COSSERAT, COSSERAT 1909a):

$$\mathcal{A}_t = \mathcal{A}_{t\beta}\mathcal{T}_\beta = \xi\mathcal{T}_1 + \eta\mathcal{T}_2 + \zeta\mathcal{T}_3 + p\mathcal{T}_4 + q\mathcal{T}_5 + r\mathcal{T}_6 \tag{24}$$

And for $b = i = x, y, z,$ one obtains the “geometric velocities”:

$$\mathcal{A}_i = \mathcal{A}_{i\beta}\mathcal{T}_\beta = \xi_i\mathcal{T}_1 + \eta_i\mathcal{T}_2 + \zeta_i\mathcal{T}_3 + p_i\mathcal{T}_4 + q_i\mathcal{T}_5 + r_i\mathcal{T}_6 \tag{25}$$

It means that all together we have 4×6 = 24 components of Cosserats deformations components which are the function of six parameters of Helmholtz group. Unfortunately, in the 1909-1935 this helpful structure of non-abelian gauge group was far to turn whole elasticity theory into the frame of gauge field theory. In the period 1909-1950 the original concept of Cosserats has steeply been forgotten.

[4] There is an anecdote relating with the notion of “wryness”. One of us (J. Badur) in December 1984 visited Technical University at Saint Petersburg. The aim of visit was to meet prof. Pavel Zhylin. The meeting takes place in Zhylin’s cathedral room accompanied by a guard from the university's security bureau with a Kalashnikov rifle on his back and a mysterious employee of the secret police. The atmosphere of the meeting was tense. When J. Badur handed over his article (PIETRASZKIEWICZ, BADUR 1983a), the police officer first checked it for a long time, page by page, before it could be handed over to Pavel Zhylin. In this situation, it was not known what to talk about, and Zhylin was not sure to whom he is talking. Then he pointed to the article ERICKSEN and TRUESDELL (1958) and asked, indicating to the first mathematical equation – what is this object? When J. Badur, without hesitation, replied that “it is a wryness connection”, the anxiety and tension subsided, and the two participants began to talk about the "intrinsic approach".

The Sudria vectors notation

Before that, in 1935 Joachim Sudria (SUDRIA 1925, 1935) a scientist related with Toulouse, developing reasoning truly in Cosserats' tradition, and having an unambiguous reference to the notion of Euclidean action[5] to witness an approach truly in the Cosserats' tradition with an unambiguous reference to the notion of Euclidean action. Having a critical treatment to the Planck concept of "smallest quantum of action" Sudria underline that after Lagrange yet Lazare Carnot was sufficiently hostile to that notion being given a priori that he deemed any proof that contained the word "force" to be absurd (CARNOT 1793).

Keeping itself the doctrine of action, Sudria, underlying the French tradition of developing of a set governing equation only on the base of variational treatment of the action, has rewritten the "intrinsic" Cosserats equation in terms of "vectors". It means, that Sudria has resigned with Lie group algebra notation: ξ_b, η_b, ζ_b $(b = t, x, y, z)$ (translational deformation potentials) and p_b, q_b, r_b ($b = t, x, y, z$) (rotational deformation potentials) and, as a form of simplification, he introduced two "vector measures": $\Upsilon_b = \xi_b d_1' + \eta_b d_2' + \zeta_b d_3'$ and $\Omega_b = p_b d_1' + q_b d_2' + r_b d_3'$ (nowadays (BADUR, PIETRASZKIEWICZ 1986, BADUR 1993b) denoted as: $\hslash_b, \ell_b$, respectively), where three base of fixed (Lagrangean) frame are: $d_1 \equiv d_x, d_2 \equiv d_x, d_3 \equiv d_z$. Notice, that the form this vector denotation is only one step to tensor notation: $\mathcal{V} = \Upsilon_b \otimes d_b'$ and $\mathcal{L} = \Omega_b \otimes d_b'$ which is commonly used in the literature (GÜNTHER 1958, KRÖNER 1960, TOUPIN 1962, MINDLIN, TRIESTEN 1962, ERINGEN, SUHUBI 1964).

As energetical partners for deformation Υ_b and curvature Ω_b Sudria defines stress: $\mathcal{E}_b = A_b' d_1' + B_b' d_2' + C_b' d_3'$ and couple $\mathcal{M}_b = P_b' d_1' + Q_b' d_2' + R_b' d_3'$ vectors, with a special definition a case when $b = t$ (time). With these definitions the four-time Cosserats equations of motion (COSSERAT, COSSERAT 1909a, §63) written for convective coordinates ϱ_b take a more compact form (SUDRIA 1935):

$$\frac{D}{D\varrho_b}\,\mathcal{E}_b = f_r, \frac{D}{D\varrho_b}\,\mathcal{M}_b + \Upsilon_b \times \mathcal{E}_b = m_r \tag{26}$$

where $(b = t, x, y, z)$ and f_r, m_r are intrinsic translational and rotational Cosserat action sources. In the above Sudria defines the divergence operator to be:

$$\frac{D}{D\varrho_b}\,(\cdot) = (\cdot)_{,\rho_b} + \Omega_b \times (\cdot) \tag{27}$$

The future of mechanics has shown that the Sudria technique via "vectors" Υ_b, Ω_b and $\mathcal{E}_b$, $\mathcal{M}_b$ does not develop more on the ground of 3D medium.

[5] J. Sudria even cite Léon Brillouin: "Among the physicists, who dares to boast that they have a clear idea of action?"

However, what is surprising, in the field of thin beam (wires, strings, rods), plate and shells, modeling with using of the vector measures of deformation and stress has becomes very popular and have been introduced quite independently in a different context (TONOLO 1930, KRAUß 1929, SYNGE, CHIEN 1941, CHEN 1944, ZERNA 1950). For the better denotation, we will call the pure deformation vector Υ_b as the Sudria vector, and the curvature Ω_b as the Darboux vector.

Let us mention, for historical precision, that Sudria vector notation has not been accepted on the ground of 3D Cosserats bodies, but within the framework of rods, plate and shells both Sudria deformation vector and the Darboux curvature vector were frequently used. One of the veteran of this treatment was Wojciech Pietraszkiewicz known from his celebrated report *Introduction to non-liner shell theory* (PIETRASZKIEWICZ 1988). Other papers by GREEN et al. (1965), SIMMONDS and DANIELSON (1972), REISSNER (1981), BASAR (1987), MAKOWSKI and STUMPF (1990), SANSOUR and BUER (1992) are examples of universality of Sudria formulation.

Remark 5. On the beginning of 60, the Sudria vector notation has become popular among the group of German scientists (Günther, Kröner, Kessel, Hermann Schaefer, Kluge, Besdo, Lippmann, as well as Eric Reissner). It was in 1958 when Wilhelm Günther introduced a complete set of vector presentation of the Cosserats continua but limited to the geometrically linear case. In that case the differences between Eulerian, Lagrangean and intrinsic description vanish and the Sudria Υ_b, Ω_b and $\mathcal{E}_b$, $\mathcal{M}_b$ becomes: $\Upsilon_i \cong \varepsilon_i + 1$; $\Omega_i \cong \kappa_i$; $\mathcal{E}_i \cong \mathfrak{B}_i$; $\mathcal{M}_i \cong \mathfrak{T}_i$. Then, the first and second Cauchy laws (eq. 26) take the vectorial form: $\partial_r\left(\sqrt{g}\mathfrak{B}_r\right) + \sqrt{g}f_0 = 0$ and $\partial_r\left(\sqrt{g}\mathfrak{T}_r\right) + e_r \times (\sqrt{g}\mathfrak{B}_r) + \sqrt{g}m_0 = 0$ – which are important in any curvilinear coordinate system with g being the measure of metrics.

Instead of fully nonlinear vectors: Υ_i and Ω_i Günther has proposed linear expressions as: $\varepsilon_i = \partial_i u + \partial_i \times \varphi$; $\kappa_i = \partial_i \varphi$ where u, φ are the vectors od displacement and rotation. Probably Günther is the first who is able to rewrite the Cosserats compatibility conditions in terms of the linear Sudria deformation and Darboux curvature vectors (GÜNTHER 1958, eq. 2) as: $\partial_i \varepsilon_j + e_i \times \kappa_j = 0$; $\partial_i \kappa_j - \partial_j \kappa_i = 0$. These are nothing else as the linearized 18 Cosserats compatibility equations. Next, if the deformations ε_i, κ_j are given as independent of Cosserats displacement and rotation then, as Günther postulate, they must define some “incompatibility objects”: $J_k^{(r)} \equiv \epsilon_{ijk}\partial_i \kappa_j$; $J_k^{(t)} = \epsilon_{ijk}(\partial_i \varepsilon_j + e_i \times \kappa_j)$ which are the measures of strain incompatibilities. Additionally, these objects must fulfil the co-called Bianchi differential identities (GÜNTHER 1958, eq. 2.16, 2.17): $\partial_i J_i^{(r)} = 0$; $\partial_i J_i^{(t)} + e_i \times J_i^{(r)} = 0$.

Generally, Günther after by developing some “primary sequence of deformation set of equations” was able to find quite similar sequence for stresses and moments. Among them he proposed 18 stress function (GÜNTHER

1958, eq. 3.11, 3.12): $\mathfrak{B}_r = \epsilon_{ijr}\partial_i \mathfrak{f}_j$ and $\mathfrak{T}_r = \epsilon_{ijr}(\partial_i \mathfrak{h}_j + e_i \times \mathfrak{f}_j)$. In this manner Günther has shown that a structure of governing equations of the Cosserats continua is the same as the general structure of the classical field theory known as the Tonti primary and dual diagrams (TONTI 1976). In the differential geometry these structures are known as the "Spencer sequences" – the complete development of their knowledge one can find in Jean-François Pommaret's works (POMMARET 2010, 2014, 2016).

Let us mention that 3D approach is consistent with Euclidean geometry therefore initial reference curvature vectors are unpresented in $\partial_i \kappa_j - \partial_j \kappa_i = 0$. In shells the situation is different from very beginning since a surface has 2D Riemannian geometry. If ℓ_α, $\alpha = 1, 2$ is the actual Darboux curvature vectors, then the curvature compatibility equations: $\ell_{1,2} - \ell_{2,1} + \ell_1 \times \ell_2 = 0$ is fully equivalent to three scalar Gauss-Codazzi equations typical for Riemannian space. According to Reissner, one can write the Darboux curvature vectors as a sum of the reference, undeformed Darboux vector $\mathring{\ell}_\alpha$ and the Reissner deformation curvature κ_α as: $\ell_\alpha = R\mathring{\ell}_\alpha + \gamma_\alpha \kappa_\alpha$ (REISSNER 1950, SIMMONDS, DANIELSON 1972, REISSNER 1974, PIETRASZKIEWICZ 1979) where deformed base is $a_\alpha = x_{,\alpha} = \gamma_\alpha d'_\alpha$.

Intrinsic versus natural approach – a time after Cosserats

In summary, let us highlight three research paradigms. From very beginning, we have three ways of describing the physics of a continuum: two of them, the description of Lagrangean (mainly for solids) and Eulerian (for fluids) belong to the natural description.

The third description that the Cosserats developed in their two monographs (COSSERAT, COSSERAT 1896, 1909a)[6] is the description of the type "intrinsic" (both from lagrangean and eulerian point of view). This description has not been taken up in the literature and contemporary plays a marginal role. Nevertheless, the continuum with the asymmetric stress tensor and couple stress has been developed since 1950 either in the description of Lagrange (LE CORRE 1965, GÜNTHER 1958, GRIOLI 1960, KOITER 1964, NOWACKI 1966) or the Euler description (BESSAN 1963, ERINGEN, SUHUBI 1964, LIPPMANN 1969, ALBLAS 1969, BADUR et al. 2015).

It would seem that the "intrinsic" description should go to the dungeons of history. And it will be forgotten once and for all. At least that's what the work of mechanics indicated: NOWACKI (1986), EHLERS et al. (2003),

[6] The paper: *Note sur la théorie de l'action euclidienne* (COSSERAT, COSSERAT 1909b) can not be treated as completly different form monography (COSSERAT, COSSERAT 1909a).

TRUESDELL (1953), REISSNER (1974), GRIOLI (1960), MAUGIN (2014), BASAR and WEICHERT (2000).

Even Truesdell and Toupin in their fundamental monography (TRUESDELL, TOUPIN 1960) on the classical field theory, have overlooked the potential power of “intrinsic description”. Probably Clifford Truesdell, writing his historical review (TRUESDELL 1960) on thin elastic bodies, was not yet familiar with the France achievements in the “intrinsic” and “moving frame” descriptions. Especially, this Truesdell’s ignorance concern to the thin shells, which was a subject of his PhD thesis in 1943. Even in the paper ERICKSEN, TRUESDELL (1958) the basic articles concerning “intrinsic approach” are not cited, what should be treated as a salient removing of “intrinsic” from mechanics.

Fortunately, the Cosserats stood on the shoulders of giants, they did not invent new abstract physics, but only unified within the basis of the 3D body, earlier concepts applied to material point mechanics, mechanics of rods and beams, mechanics of plates and shells. This attempt to a unification, important for mechanics, is extremely valuable for the entire physics of the continuum, as it was noticed by the American researcher E. Wilson as early as 1913 (WILSON 1913). Today, the same attempts to unify different interactions are made within gauge field theory (MEISSNER 2013, HEHL 2017). The way of searching for new field theory models is the same as that established by the Cosserats – one needs to find a set of equations of motion and laws of currents conservation based on action:

$$\delta\mathcal{A} = \delta \int_{t_A}^{t_B} \left(\iiint_S W \; da\, db\, dc\right) dt = \delta \int_{t_A}^{t_B} \left(\iiint_S \Omega \; dxdydz\right) dt \qquad (28)$$

where W, Ω are the volumetric energy densities in the Lagrange and Euler descriptions, respectively (COSSERAT, COSSERAT 1909a). Unfortunately, apart from very general formulas for the Least Action Principle, the Cosserats have not provided any more information on possible constitutive equations, finishing their considerations on possibility of splitting kinetic energy and deformation energy. The intrinsic deformations in Eulerian description are not defined therefore the Eulerian volumetric densities Ω in eq. 28 is not written explicitly. Only in the first monography: *Sur la theorie de l'elasticite* (1896) Cosserats proposed an explicit form of lagrangean $W = W(\mathcal{V}^2) = W(E)$ consistent with the «constrained rotation» restrictions. In the above, the Lagrangean finite deformation tensor is defined to be: $2RER^T = (\mathcal{V}^2 - 1)$ (COSSERAT, COSSERAT 1896, 1909a, 1909b).

Concept of compensating potentials

Probably Herman Weyl was first who introduced in 1918 to the field theory a notion of gauge transformation as transformation related with a local group of symmetry. He has attempt to unify gravity and electromagnetism as the unifying two groups: Poincare PL(14) and unitary $U(1)$. Weyl clarified that the electromagnetic field is intimately related to local internal symmetries, under the $U(1)$ group that act on the 4-spinorfields of charged matter. The group $U(1)$ is abelian group where algebraic nonlinearities are abandoned. In the mid 1950's Yang and Mills, dealing with nuclear strong interactions, further explored the notion of gauge symmetries in field theories going beyond the $U(1)$ group to include non-Abelian Lie group $SU(2)$. Then non-Abelian algebraic nonlinearities – similar to well-known $SO(3)$ nonlinearities, have been appeared. In 1956 Utiyama reinterpreted gauge fields as "compensating potentials" of all semi-simple Lie groups including the Lorentz group.

Then the idea of gauging appears as a concept of mathematical description based on the technique of localization, some global symmetry group of the field theory, introducing a new interaction described by the gauge potential. The compensating field – mathematically is nothing else as a scalar or vector with $n-$values within the algebra of local group, where n is the dimension of Lie algebra. The main postulate of gauging was to introduce the compensating fields in a such manner that makes it possible for the fields and matter action to be locally invariant under the symmetry group. This rule of local invariance as a result leads to a concept of minimal replacing of "classical derivative" into "covariant gauge derivative". Simultaneously, owing the works of Trautman (MEISSNER 2013), developed a clear differential geometrical interpretation of the gauge potential as the connection of the fiber bundle, which is the manifold obtained from the base space-time manifold and the set of all fibers. These are attached at each space-time point and are the vector, tensor or spinor spaces of representation within n-parametrical algebra of the local symmetries. What is important in this geometrical interpretation, that the imposition of local symmetries implies that the geometry of the fiber bundle is non-Euclidean, and the gauge field strengths tensor is the two-form of curvatures of such a manifold (POMMARIET 1989, POMMARET 2016, DE LEÓN et al. 2021).

The gauge field theory can be also interpreted as some intrinsic description where intrinsic space of Lie algebra is introduced by "moving frame" of a symmetry group. Thomas Craig, an American physicist, was a pioneer in applying the Cosserats moving frame to the description of space-time deformation of gravitation continua (CRAIG 1898). Such w concept of gravitation gauge potential was in contrary to Władysław Gosiewski's concept of gravitation deformation metrics (GOSIEWSKI 1877, BADUR 2022).

The Gosiewski model of gravitation has used the space metrics g_{ij} as a gravitation potential where continuum deformation has been induced by insertion of a massive body. Therefore, in contemporary field theory there are exist two quite different models of gravitation: the first is based on space--time metric: $g_{\mu\nu}$ as the gravitational potential (a fundamental unknown) and second is based on gauge compensating field $\mathcal{A}$ as fundamental gravitational field (BADUR 2022).

Élie Cartan (CARTAN 1925), the French geometer of Lie group of moving frame, was the next who adopted the Cosserats intrinsic description to describing of gravitation potential adopted to any space-time relativity conditions. Unfortunately, the Cartan gravitation model has been development in the language of modern differential geometry, therefore it cannot be appreciated as a fully physical model. Only in 1965, HEHL and KRÖNER (1965a, 1965b) by precise consideration of group arguments in the Cosserats vision of continuum were able for noting the rich possibility of gravitation modeling though include the action of distributed couples along with more classical contact forces. In contrary to common general relativity model, the model of gravitation based on the Cosserats, has contains a new additional physical quantities like "spin" and "torsion" (HEHL 1973, HEHL, OBUKHOV 2007, LAZAR, HEHL 2010). Hehl notices that a fundamental difference between general relativity and the Cartan model of gravitation lies on the fact that the Cartan model is completely the gauge field theory only based on an algebraic nonlinearity, whereas the general relativity is based on geometric nonlinearity (HEHL 2017).

Concept of anholonomy frame

The group of Italian geometers, among them T. Levi-Civita and Gregorio Ricci-Curbastro, have developed a concept of arbitrary moving frame with $a, b, c \ldots = 1, 2, 3, 4, 5, \ldots, n$ frame vectors e_a, called the anholonomy base[7]. The rule of differentiation in "anholonomy world" was not a simple one, firstly it must be defined the differentiation operator as: $\partial_a = \frac{\partial}{\partial \varrho_a} = R_{ai} \frac{\partial}{\partial x_i} = R_{ai} \partial_i$ where R_{ai} anholonomy coefficients. This anholonomic derivative leads to the anholonomy base differentiation to be: $\partial_a e_b = \gamma_{abc} e_c$. The coefficients γ_{abc}, called the Ricci rotation coefficients (SCHOUTEN 1954), have no any similarity

[7] In the gravitation this base is called *Vierbeinen* (HEHL 1973). If one retains the idea of a difference then one may also remark that the first theory of gravity makes recourse to only Riemann space, therefore, a space without torsion; the second one makes recourse to a space with torsion, in the sense of Élie Cartan. However, Hehl has proof, that a space with torsion may be represented on a space without torsion by means of the absolute differential calculus and the Ricci rotation coefficients (HEHL, OBUKHOV 2007, SIMON, DELL'ISOLA 2017, 2018a, 2018b).

to the Christoffel symbols, since they are defined without any reference to the space metrics as: $\gamma_{abc} = R_{ak}R_{bj}\partial_k R_{cj}$ where $i, j, k = 1, 2, 3$ are related with the holonomic system of coordinates. Geometers Cartan, Ricci, Schouten and others have shown that even if the holonomic space is a Euclidean (no curvature and torsion) the anholonomy system in general possesses both curvature and torsion. Élie Cartan (CARTAN 1925, p. 367) defined the components of torsion $\mathfrak{t}_{abc}$ and curvature $\mathfrak{r}_{abcd}$ by the formulas:

$$\partial_a(\partial_b f) - \partial_b(\partial_a) f = \partial_{[a}\partial_{b]} f = (\gamma_{abc} - \gamma_{bac})\partial_c f = \mathfrak{t}_{abc}\partial_c f \tag{29}$$

$$\mathfrak{r}_{abcd}(\gamma) = \partial_d\gamma_{abc} - \partial_c\gamma_{abd} - \gamma_{abe}\mathfrak{t}_{ecd} + \gamma_{ead}\gamma_{ebc} - \gamma_{eac}\gamma_{ebd} \tag{30}$$

Briefly, Ricci's model of immersed anholonomic manifold, in its fundamental formulas, recalls, at the same time, the formulas of Cosserats which are only limited to the cases when the anholonomic indices are low: $a, b, c, d = 1, 2, 3$ and no more. It was Thomas Craig who was the first to introduce a model of gravitation with the indices $a, b, c, d = 1, 2, 3, 4$ (CRAIG 1898) and his model of gravitation has no relation with any Riemannian metrics taken as the gravity potentials. Craig's model was developed in USA by Wilson, Tolman, Murnaghan, G.N. Lewis, and was discussed also by Whitehead in his relational model of gravitation (BADUR 2022). However, in the theory of groups, as proposed by CARTAN (1935), SCHOUTEN (1954), KRÖNER (1960), may be a theory of continuum with torsion; one may thus have, in the theory of Ricci coefficients, everything that one has in that of Riemann spaces, generalized by the appearance of torsion.

One information is an important for correct understanding the differential tools related with a moving frame. For example, how to calculate the operation of divergence if, according to Cosserats, we use the rotated second Piola-Kirchhoff stress: $S' = RSR^{-1}$ (COSSERAT, COSSERAT 1896, eq. 22). Or, similarly, how to write explicit component form of Sudra vectors (eq. 26). As Angelo Tonolo has pointed us (TONOLO 1930), the anholonomy base is differentiated by the Ricci rotation coefficients, then the divergence of the rotated second Piola-Kirchhoff stress should be calculated with two not one Ricci rotation coefficients: $\mathrm{Div}'(S') = (\partial_b S_{ab} + \gamma_{cda}S_{cd} + \gamma_{dca}S_{cd})e_a$ (TONOLO 1930, eq. 1) or $\mathrm{Div}'(S') = \partial_b S_{ab}e_a + S_{ab}\ell_a \times e_b + S_{ab}\ell_b \times e_a$ if we develop the Sudria equation (27).

The fruitful analogy

The classical Cauchy continuum model considers material continua (fluids, solids, grains, foams, etc.) as a collection of simple particle-continua with points having three displacement-degrees of freedom, and the response of a material to the deformation is characterized by a symmetric Cauchy stress tensor. Such mathematical model is governed by the equation of motion which is identified with the balance of linear momentum. The balance of moment of momentum, established by Euler in 1752 (EULER 1752) for a single rigid body is automatically fulfilled in the Cauchy model of continuum. In the whole XIX century, developing of mechanics of rigid body, thin beams and thin shells lead to increasing a role of finite rotations in description of motion and deformation of solid continua. In 1884 Hess discovered (HESS 1884) quite unusual analogy between mechanics of thin Euler-Kirchhoff beam model and mechanic of rigid body. When he has changed the time (t) coordinate in six rigid body equation of motion into length coordinate (s) in the six equation of a thin beam statics, then it appears that these equations have the same structure. Hass called this phenomenon as "kineto-static analogy".

Using this analogy in a little extended manner, where time and space coordinates play the same: "time-governed" role, Aron in 1874 (ARON 1874), constructed from very beginning a deformable surface model where three parameters: two surface convective coordinates and time coordinate were used for construct a "surface-time analogy". In constructing the model of surface Aron was under influence of Sophie Germain, which has treated the plate as to be a crossing two elastic, flexible, thin, Euler beams. In Aron's analogy, the time coordinate can be replaced mutually with the surface convective (Gauss) coordinates and three translations, and three rotations of any surface points become the basic unknowns.

Next, Aron's paper was the subject of critical analysis by LOVE (1888) who, first of all, made linearization and next elimination of rotations as independent variables. To eliminate rotations Love used geometrical constrains, nowadays known as "Kirchhoff-Love constrains". In this approach, rotation of beams and shells, even if huge, are eliminated and become hidden parameters. What important, concept of couples, would be further used since it has relation with engineering practice. Günther says that: the doors for 3D+time continua were opened (GÜNTHER 1961). The solid body can be described as rigid particles continuum with four-time parametrization, since tree spatial parameters acts in the same manner as time. François and Eugène Cosserats were be first in discovery of four-time property of Euclidean time-space and first in introducing of 4-time relativity. The pioneers of using intrinsic formulation before Cosserats are listed as: EULER (1752), LAGRANGE (1762), PIOLA (1833), DELL'ISOLA et al. (2015), KIRCHHOFF (1852), SKIBA (1874), FRANKE (1889), BASSET (1894, 1895).

In the literature there are many examples of using Hess' analogy to construct a mathematical model of shell surface with displacement and independent or dependent rotations. These examples nowadays are excellent patterns for developing the string theories in the quantum field modeling. Let us mention the papers of: HENCKY (1915), WEATHEBURN (1927), KRAUß (1929), VALID (1979), BASAR (1987), MAKOWSKI and STUMPF (1990), CHRÓŚCIELEWSKI et al. (1992, OPOKA and PIETRESZKIEWICZ (2004) and others. Even realistic, very practical models of rods, strings and lines have the base located within intrinsic approach see: DANIELSON and HODGES (1984), HODGES (1990), DILL (1992), O'REILLY and TURCOTTE (1997), CRISFIELD and JELENIĆ (1998), GRUTTMAN et al. (1998), LUO (2010).

The concept of constrained rotations (1896)

The first attempts to Cosserats model of continuum have really been started in his pioneering paper (COSSERAT, COSSERAT 1896). The basic variables of the Cosserats continuum are six unknown functions: for a solid there are the displacement and finite rotation of the particle of the continuum, and for fluids there are the material velocity and vorticity of the fluid element. Mobile reper $\mathbb{R}' = \{x', d'_x, d'_y, d'_z\}^T$ to which we refer the description of the motion moves in relation to the output reper $\mathbb{R} = \{x, d_x, d_y, d_z\}^T$ (see: Fig. 2) by displacement $u = x' - x$ and rotation $d'_a = Rd_a$ $(a \equiv x, y, z)$. In general, the rotation R is arbitrary, but in the concept of constrained rotation, the rotation ceases to be an independent variable and becomes dependent on displacements in such a way that $R \equiv \mathcal{R}$ where the rotation $\mathcal{R} = \mathcal{R}(u, \nabla u)$ is determined from the polar decomposition of the deformation gradient tensor:

$$F = \mathcal{V}\mathcal{R} = \mathcal{R}\mathcal{U} = \mathrm{Grad}\, u = u \otimes \nabla \tag{31}$$

Such a concise polar decomposition, devised precisely for the purpose of the concept of bound rotation, was devised by the Cosserats brothers as early as 1896 (COSSERAT, COSSERAT 1896, eq. 97). To introduce this condition in a mathematically unambiguous way, they write: $\mathcal{V} = \mathcal{V}^T$ (COSSERAT, COSSERAT 1896, eq. 99), i.e. they put three additional scalar conditions on the side components $\mathcal{V}$.

So, the idea is as follows. Instead of the classical description within a fixed Cartesian frame of reference (Lagrangean or Eulerian), we can introduce new kinetics of description using a moving reper $\mathbb{R}$, which rotates in a manner determined by displacements so as to maintain the symmetry of the objects used. In this approach, there are no more torque stresses and angular momentum equations. However, we use explicitly, a rotating reference trihedron in order to more easily separate the insignificant quantities of deformation from the enormous rotations that contribute nothing to the internal energy. This thought guided Love in constraining Aron's 2D+time model or Pietraszkiewicz when writing about the new role of finite rotations in the description of the 3D deformation of the Cauchy continuum (PIETRASZKIEWICZ, BADUR 1983a).

Writing the condition of constrained rotation $\mathcal{V} = \mathcal{V}^T$ in a way that uses the Cosserat notation, we have:

$$\mathcal{V} = \begin{pmatrix} \xi_1 & \xi_2 = \eta_1 & \xi_3 = \zeta_1 \\ \eta_1 = \xi_2 & \eta_2 & \eta_3 = \zeta_2 \\ \zeta_1 = \xi_3 & \zeta_2 = \eta_3 & \zeta_3 \end{pmatrix} = \mathcal{V}^T \tag{32}$$

We use Cosserats' notation here, because we remember that the transition of "torsor" to "tensor" radically changes the doctrine of description within the mobile frame. It should be emphasized here that the mechanics of flexible rods and thin shells consistently apply the description in the mobile reper or in some other "intrinsic" base of coordinates. The most famous example is the Cosserats surface statics equations derived in 1874 by Aron (ARON 1874) and repeated independently by REISSNER (1950, 1974). It is different in 3D+time mechanics, where either a purely Lagrangean or purely Eulerian description is used, and internal objects are given a tensor character (e.g. translational deformation tensor and rotational deformation tensor).

Let us return in this paragraph to the revaluation of the main achievements of Cosserats' work entitled: *Sur la theorie de l'elasticite* (COSSERAT, COSSERAT 1896). This work is wrongly overlooked as contributing nothing to the Cosserats model. Even the eminent researcher and expert on the Cosserats model, Prof. Jean-François Pommaret, forgot about it. In 1995, Pommaret heard of unknown results and without delay came from Paris to Poitiers by a bicycle to meet one of these authors. His surprise, when he saw 18 equations of continuity of deformations (COSSERAT, COSSERAT 1896, eq. A, p. 186), was complete, he decided to make a special article informing about: "Cosserats secrets" (POMMARET 1997). Thus, in the literature, prof. Pommaret is known also as the specialist form Cosserats' live and Cosserats' achievements. Nevertheless, probably, the best scientific analysis of Cosserats efforts are given by WILSON (1913)[8] and NEFF (2019)[9].

[8] Edwin Bidwell Wilson (1879–1964) was an American mathematician-physicist who had been a PhD student of J. W. Gibbs at Yale, and became Professor of mathematics first at M.I.T and then at Harvard. He was co-authored a book on vector analysis with Gibbs (first edition, 1901, then several further editions). He proposed in 1911, the first model of general relativity which explicitly fulfils the invariance under the Lorentz group of symmetry. He proposed also a hyperbolic model of space-time – which is consistent both with electrodynamics and gravitation.

[9] Patrizio Neff (1999) An outstanding researcher concerning the Cosserats model. He is a chef Professor at Lehrstuhl fur Nichtlineare Analysis, Universität Duisburg-Essen.

In short, the work of the Cosserats from 1896 concerns the Cauchy continuum, but it has a novelty in the form of writing kinematic equations, constitutive equations and equations of motion which are referred to the moving base of the d'_a trihedron – this motion is originally marked as a mapping of the reper: $Mxyz \rightarrow M'x'y'z'$. This notation is slightly more complete and concise than the notation: $\mathbb{R} \rightarrow \mathbb{R}'$, because it informs that the differentiation operation with respect to the moving coordinates $x'y'z'$ will be performed using motionless coordinates xyz first by “pull-back” of the object from the primed to the non-primed system, then by differentiation with respect to the non-primed coordinates and then by "push-forward" the result to the primed system[10].

Cosserats write the vector of displacement using in their model once a reference to a stationary base and once to a mobile base (COSSERAT, COSSERAT 1896, eq. 94) $u = ud_x + vd_y + wd_z = u'd'_x + v'd'_y + w'd'_z$. The rotation of bases is described explicitly by the rotation tensor: $d'_b = R_{b'a}d_a$ where $b' = x', y', z'$ and $a = x, y, z$. Its nine components are directional cosines of the projections of the corresponding base vectors; they can, due to the orthogonality of rotation $R^{-1} = -R^T$, be written using a finite rotation vector or three Euler angles – measured from $Mxyz$ when we push an object or measured from $M'x'y'z'$, when we pull an object. Cosserats decide to describe the rotation using a unit rotation vector e with the same components (l, m, n) relative to d'_b and d_a. So, $e = ld'_x + md'_y + nd'_z = ld_x + md_y + nd_z$. The measure of the magnitude of rotation is the angle ω. Having (l, m, n, ω) and remembering that $e^2 = 1$ d'Olinde Rodrigues (see: BADUR 2022) proposed four quantities (COSSERAT, COSSERAT 1896, p. 124):

$$\lambda = l \sin\frac{\omega}{2}, \mu = m \sin\frac{\omega}{2}, \nu = n \sin\frac{\omega}{2}, \rho = \cos\frac{\omega}{2} \tag{33}$$

which allow to represent the components of the rotation tensor briefly as (COSSERAT, COSSERAT 1896, p. 125)

$$R_{b'a} = \begin{pmatrix} \rho^2 + \lambda^2 - \mu^2 - \nu^2 & 2(\lambda\mu - \nu\rho) & 2(\lambda\nu + \mu\rho) \\ 2(\lambda\mu + \nu\rho) & \rho^2 + \mu^2 - \lambda^2 - \nu^2 & 2(\mu\nu - \lambda\rho) \\ 2(\lambda\nu - \mu\rho) & 2(\mu\nu + \lambda\rho) & \rho^2 + \nu^2 - \lambda^2 - \mu^2 \end{pmatrix} \tag{34}$$

This is an extremely comfortable character, used in numerical practice. This character is also used by the Cosserats in their 1909 monograph.

[10] There are in the literature a lot of extensions of deformable moving reper concept. Especially, if continuous crystal defects need to be described – LE and STUMPF (1998), CAPRITZ and VIRGA (1994), FOREST et al. (1997), NEFF (2006), FOREST and SIEVERT (2006), PAPENFUSS and FOREST (2006), CLAYTON (2022) are among them.

However, as indicated (BADUR, CHRÓŚCIELEWSKI 1983, PIETRASZKIEWICZ, BADUR 1983a), this form is not very convenient for differentiating and integrating the rotation tensor, hence different representations are used in the literature. The Cosserats, apart from the representation of Rodrigues, which in itself does not have much reference with the representation of Euler, do not use other solutions such as: Leibniz's monad, Hamilton's quaternion, Grassmann extensions, Dirac spinors, etc. (BADUR 2022).

The mechanics of the Cauchy continuum, as presented in the base of the constrained mobile reper $M'x'y'z'$, has required additional operations from the Cosserats, translated in the language known from description of the fixed $Mxyz$. There are new objects unknown in Cauchy mechanics, mainly we mean the differences resulting from the differentiation of the fixed base d_a and the mobile base d'_b. The base d_a is a constant one, hence only the physical components of e.g. stress tensors must be differentiated.

As a new object the Cosserats have introduced the rotated Piola-Kirchhoff stress tensor (COSSERAT, 1896, eq. 22) which is:

$$S' = RSR^{-1} = P_1 d'_x \otimes d'_x + P_2 d'_y \otimes d'_y + P_3 d'_z \otimes d'_z + U_1\left(d'_x \otimes d'_y + d'_y \otimes d'_x\right) + U_2(d'_x \otimes d'_z + d'_z \otimes d'_x) + U_3\left(d'_y \otimes d'_z + d'_z \otimes d'_y\right) \tag{35}$$

with six components $P_1, P_2, P_3, U_1, U_2, U_3$ (nowadays $S_{ab} = S_{ba}$) – numerically equal to the components of the un-rotated second Piola-Kirchhoff tensor S. Occurring in the equation of motion S' requires knowledge of formulas for differentiation of the mobile base. You can use generalized formulas on the Frenet trihedron or Poisson formulas for differentiation of the mobile reper after time:

$$d'_{x,t} = qd'_y - rd'_z, d'_{y,t} = rd'_z - pd'_x, d'_{z,t} = pd'_x - qd'_y \tag{36}$$

Which can be shortly written as: $\partial_t d'_i = d'_{i,t} = \ell_t \times d'_i$. To be precise, the time denotation should be t' instead of t. This is the distinction the Cosserats make in the next chapter (COSSERAT, COSSERAT 1896, § 41), in which they deal with the dynamics of the surface inserted into the movable reper. Then denote time as a separate parameter ρ_1 and assign it the role of time ("quand ρ_1 varie scule et joue le rôle du tempes"). This confirms our thesis that the Cosserats is building a 4-time intrinsic continuum where space coordinates are treated to be time-like parameters (BADUR, CHRÓŚCIELEWSKI 2015).

The Poisson parameters p, q, r occurring in (eq. 36)[11] depend entirely on the time derivatives of the rotation tensor $R_{b'a}$. Hence, treating the coordinates x', y', z' as three additional times, the Cosserats obtain Poisson curvature parameters for: $x' \to p_1, q_1, r_1$, for $y' \to p_2, q_2, r_3$, and $z' \to p_3, q_3, r_3$, respectively. The determination of these curvature components as a function of the Euler angles or the finite rotation vector function has been the subject of numerous works, because it is an important element of kinematics unknown for researchers familiar only with kinematics of the Cauchy continuum. We will not deal with this element by referring the reader to works on the theoretical mechanics of the material point where numerous formulas for p, q, r are given as a function of, for example, Euler angles, or to the work of the mechanics of rods and thin shells[12].

Remark 6. However, what is important from the point of view of the concept of a constrained mobile reper is the time curvature parameters p, q, r and space curvature parameters p_a, q_a, r_a should be represented by derivatives not of the rotation tensor but of the translational deformation measures. This is necessary because in the calculation procedure there are no turnovers, but only combinations of derivatives expressing them. Such a missing formula for the parameters p_a, q_a, r_a, $a = x', y', z'$ for the 3D continuum was given in 1983 by PIETRASZKIEWICZ and BADUR (1983a) in terms of the Green strain tensor components and in paper (BADUR 1993b) in terms of $\mathcal{V}$ components. It should be added here that the parameters p_a, q_a, r_a define a curvature vector $\ell_a = p_a d'_x + q_a\, d'_y + r_a\, d'_z$, which after Cosserats', is called the Darboux curvature vector. It allows us to briefly write equations (36) as: $\partial_b d'_a = d'_{a,b} = \ell_b \times d'_a$.

Let us emphasize that formulas for Darboux vectors in terms of deformations and their derivatives for two- and one-dimensional continuums are more complicated than for 3D one. An example is Love's paper (LOVE 1888) on constrained Aron's curvature vectors ℓ_β $(\beta = 1, 2)$ for curvilinear coordinates ϱ^β on two-dimensional surface to get constrained rotations as independent variables. Love eventually obtained linearized expressions for $\ell_\beta = \ell_\beta(u; u_{,\beta}; u_{,\beta\gamma})$ as a function of the displacement of surfaces and their first and second derivatives, and his model records the equation of motion (equilibrium) in the rotated basis. In this sense, the name

[11] It is a tradition of using the same letters od denotations in the whole literature. Proposed by Poisson letters p, q, r are applied by: KIRCHHOFF (1859, 1883), LECORNU (1880), HESS (1884), FRANKE (1889), DARBOUX (1890, 1900), CARTAN (1925), SUDRIA (1925), CESARO (1926), DELENS (1927), CESARO (1926) and others.

[12] These relations should be useful do numerical simulations, especially in Finite Element and Finite Volume Method. Let us mention papers by: SANSOUR (1998b), GRUTTMAN et al. (1998), WIŚNIEWSKI (1998), NADLER and RUBIN (2003), CHRÓŚCIELEWSKI et al. (2010), GRUTTMANN et al. (1989).

”Kirchhoff-Love shell theory” is a bit misleading. Kirchhoff, continuing Poisson's metric approach, develops the theory in a natural, metric approach, while Love, following Aron, develops an intrinsic approach. Therefore, for the sake of accuracy, we should speak of the ”Aron-Love” intrinsic model and the ”Poisson-Kirchhoff” natural model.

The difference between these models is fundamental, just as the difference between Hehl's and Einstein's gravity model is fundamental. In the years 1970-2000, the description of the intrinsic type, thanks to the development of computer techniques, became popular and effective. Jacek Chróścielewski (BADUR, CHRÓŚCIELEWSKI 1983, CHRÓŚCIELEWSKI et al. 1992, 2004, 2010) and Carlo Sansour (SANSOUR, BUER 1992, SANSOUR 1998a, 1998b, SANSOUR, SKATULLA 2008) made significant contributions to this field. Another important applications are made due to efforts of ATLURI and CAZZANI (1995), DE BORST (1991) and RUBIN (2000).

Remark 7. In the work under discussion, the Cosserats devote much space to the equations of continuity of deformations derived as part of the ”natural” approach by Berré de Saint-Venant (see POINCARÉ 1892). Cosserats ask whether the intrinsic approach with constrained rotation can derives the same “natural” strain continuity equations. This question was very occupied the Italian geometers like BELTRAMI (1871, 1911) and CESARO (1926). Even later, the Italian mathematicians TONOLO (1930), FINZI (1932) and PASTORI (1934) developed the intrinsic approach based on constrained finite rotation. Complete set of governing equation based on both polar decomposition $F = \mathcal{VR} = \mathcal{RU}$ proposed in his valuable paper SIGNORINI (1943). Modern statement of intrinsic formulation of 3D media has been developed by FERRARESE (1959, 1971, 1972, 1976), GRIOLI (1960, 1968), BESSAN (1963), STASI (1976) and CAPRIZ and PODIO-GUIDUGLI (1977). Revalorization of Italian works has been made by ZHONG-HENG (1963) and ZHOUNG-HENG and DUBEY (1983).

Remark 8. In 1896 the Cosserats the main goal was to obtain the balance of linear momentum in the rotated frame. Let note, that the Cosserats presents the correct panorama of whole set of balances, both; in Eulerian natural and Lagrangean natural descriptions. They introduce by a variational treatment the first and second Cauchy law: $\operatorname{div} t + \rho b = 0$ and $t = t^T$, respectively (COSSERAT, COSSERAT 1896, eq. 24, 25) within the Eulerian description.

Next the Cosserats propose to pull-back the Cauchy stress tensor to the form of first Piola-Kirchhoff (COSSERAT, COSSERAT 1896, eq. 36): $P = \mathcal{J}\, t\, (F^{-1})^T$ where $\mathcal{J} = |\det F| = \sqrt{\mathcal{V}^T\mathcal{V}} = \rho_0/\rho$ and $F = \operatorname{Grad} x$ is the gradient of deformation. Then changing description to natural lagrangean, the Cosserats transform the first and second Cauchy law into: $\operatorname{Div} P + \rho_0 b = 0$ and $PF^T = FP^T$, respectively (COSSERAT, COSSERAT 1896, eq. 34 and 37). They prove that:

$P = \frac{\partial W}{\partial F}$ and $S = \frac{\partial W}{\partial E}$ where $2E = F^T F - 1$ is the Lagrange deformation tensor and S is the second Piola-Kirchhoff stress tensor. By finding that: $P = FS$ (COSSERAT, COSSERAT 1896, eq. 33) Cosserats are able to write the first and second Cauchy law to be (COSSERAT, COSSERAT 1896, eq. 38):

$$\mathrm{Div}(FS) + \rho_0 b = 0 \text{ and } FSF^T = FS^T F^T \tag{37}$$

Next Cosserats propose to constitute the Cauchy stresses by the fully nonlinear relation (COSSERAT, COSSERAT 1896, eq. 61, 62):

$$t = \mathcal{J}\, F \frac{\partial W}{\partial E}\, F^T \tag{38}$$

Having the first and second Cauchy laws in the both natural descriptions, Cosserats next, introduce the concept of “rotated second Piola-Kirchhoff stress”: $S' = RSR^{-1}$ (see eq. 35. above). Then using $F = \mathcal{V}R$ Cosserats transforms $\mathrm{Div}(FS)$ into $\mathrm{Div}'(\mathcal{V}S')$ (COSSERAT, COSSERAT 1896, p. 192) what finally leads to the first and second Cauchy laws written within moving frame:

$$\mathrm{Div}'(\mathcal{V}S') + \rho_0 b' = 0, \mathcal{V}S'\mathcal{V}^T = \mathcal{V}(S')^T \mathcal{V}^T \tag{39}$$

Since Cosserats denotes $\mathcal{V}S'$ to be:

$$\mathcal{V}S' = \begin{matrix} A_{x'} d_1' \otimes d_1' + B_{x'} d_2' \otimes d_1' + C_{x'} d_3' \otimes d_1' + \\ A_{y'} d_1' \otimes d_2' + B_{y'} d_2' \otimes d_2' + C_{y'} d_3' \otimes d_2' + \\ A_{z'} d_1' \otimes d_3' + B_{z'} d_2' \otimes d_3' + C_{z'} d_3' \otimes d_3' \end{matrix} \tag{40}$$

They obtained the first Cauchy law in the rotating base, expressed by components as (COSSERAT, COSSERAT 1896, eq. 100):

$$\left(\begin{matrix} \frac{\partial A_{x'}}{\partial x} + q_1 A_{z'} - r_1 A_{y'} + \frac{\partial B_{x'}}{\partial y} + q_2 B_{z'} - r_2 B_{y'} + \\ \frac{\partial C_{x'}}{\partial z} + q_3 C_{z'} - r_3 C_{y'} + \rho_0 X' \end{matrix}\right) d_1' +$$

$$\left(\begin{matrix} \frac{\partial A_{y'}}{\partial x} + r_1 A_{x'} - p_1 A_{z'} + \frac{\partial B_{y'}}{\partial y} + r_2 B_{x'} - p_2 B_{z'} + \frac{\partial C_{y'}}{\partial z} + \\ r_3 C_{x'} - p_3 C_{z'} + \rho_0 Y' \end{matrix}\right) d_2' +$$

$$\left(\begin{matrix} \frac{\partial A_{z'}}{\partial x} + p_1 A_{y'} - q_1 A_{x'} + \frac{\partial B_{z'}}{\partial y} + p_2 B_{y'} - q_2 B_{x'} + \frac{\partial C_{x'}}{\partial z} + \\ p_3 C_{y'} - q_3 C_{x'} + \rho_0 Z' \end{matrix}\right) d_3' = 0$$

And corresponding stress boundary condition (COSSERAT, COSSERAT 1896, eq. 101). The second Cauchy law (19) is fulfilled automatically since the rotated second Piola-Kirchhoff stress S' are expressed in the function of the Lagrange strain tensor E only. In the literature of 3D continuum, there is no examples of explicit using of eq. (39), however in the low dimensional continua like rods or shells we have a lot of examples for the numerical treatment, for instance: HAY (1942), KLINGER (1942), SIMO (1992), STEINMANN (1994), IBRAHIMBEGOVIC (1994), ARMERO and ROMERO (2003).

Remark 9. There is known that the most exhaustive revalorization of the Cosserats paper on "constrained elasticity" (COSSERAT, COSSERAT 1896) was made by Antonio Signorini (SIGNORINI 1943). Signorini in his paper has no any reference to the later Cosserats monography (COSSERAT, COSSERAT 1909a), since he is only interested in developing of the 1896-Cosserats ideas. He notices that an influence of this paper onto the while Italian school of mechanics in Palerno and Bologna is evident and impressive. Signorini recalls the achievements of: Burali-Forti, Marcolongo, Burgatti, Cisotti, C. Tolotti, V. Volterra, G. Grioli, B. De Finetti, T. Boggio, Crudeli, Cesaro, D. Bonvicini, I. Gasperini, Almansi, and others.

Considering the natural Lagrangean and Eulerian descriptions Signorini has repeated the Cosserats. Signorini does not develop the moving frame in the sense of the Cosserats, since, he decides to use another polar decomposition then the Cosserats. He takes: $F = \mathrm{Grad}\, x = RU$ (left decomposition; $\alpha = \alpha_\rho \alpha_\delta$ in original notation; SIGNORINI 1943, eq. 8). His starting point is similar: in the Eulerian description first and second Cauchy law are: $\mathrm{div}(t) + \rho b = \rho a$, $t = t^T$, respectively (originally: $k(F - a) = \mathrm{grad}_P \beta$, $\beta = K\beta$; SIGNORINI 1943, p. 103 – β – omografia euleriana di tensione). Next by defining $P = \mathcal{J} t (F^{-1})^T$ he obtains the Kirchhoff stress form of equation: $\mathrm{Div}(P) + \rho_0 b = \rho_0 a$ (SIGNORINI 1943, p. 105) and further, by introducing "omografia lagrangiana di tensione": $S = \mathcal{J}\, F^{-1} t (F^{-1})^T$ (second Piola-Kirchhoff stress tensor) Signorini is able to write: $\mathrm{Div}(FS) + \rho_0 b = \rho_0 a$ (SIGNORINI 1943, p. 105-107).

Nest, using $F = RU$, Signorini defines the stretched second Piola-Kirchhoff tensor denoted by him as "δ " (SIGNORINI 1943, p. 107):

$$t = \mathcal{J}^{-1} F S F^T = R(\mathcal{J}^{-1} U S U) R^T = R \delta R^{-1} \tag{41}$$

Which is quite different from rotated second Piola-Kirchhoff $S' = RSR^{-1}$ (see eq. 35 above). Signorini proved next that stretched second Piola-Kirchhoff is equal to: $\delta = \mathcal{J}^{-1} S(1 + 2E) = \mathcal{J}^{-1}(1 + 2E)S$. Unfortunately, the measure δ is does not yet used by Signorini in the first Cauchy law. On the domain of the shell modelling this object was introduced by PIETRASZKIEWICZ (1979, 1988).

Remark 10. Probably, the best result in developing of the Cosserats'1896, Signorini has obtained in expressing of the Cosserats strain compatibility equations (see eq. 1). He found that equation (9) for $\beta = 4, 5, 6$ (rotational part) will be better to write in terms of left Darboux curvature vector $\ell_i = R\mathscr{k}_i$ $(i = x, y, z)$ (see Sudria's $\Omega_b = p_b d_1' + q_b d_2' + r_b d_3' \equiv \ell_b$). The left Darboux curvature vector is defined as an axial vector taken on:

$$\mathscr{k}_i = \left[R^{-1}R_{,i}\right]_\times = 2\vartheta\left[q_{,i} + \left(q_{,i}\right) \times q\right], \vartheta = (1 + q^2)^{-1} \tag{42}$$

If a rotation matrix is represented by the "rotation vector" $q = e \operatorname{tg} \omega/2$ in the form: $R = \vartheta[(1 - q^2)1 + 2q \times 1 + 2q \otimes q]$. As Signorini show via differentiation of polar decomposition one obtains $\mathscr{k}_i$ in terms of U and its derivatives as (SIGNORINI 1943, eq. 73):

$$\mathscr{k}_i = \mathcal{J}^{-1} U \left[\left(U_{,i} U\right)_\times + \frac{1}{2} \operatorname{rot}(U^2 e_i)\right] \tag{43}$$

or, explicitly, in components as (PIETRASZKIEWICZ 1979):

$$\mathscr{k}_i = \mathscr{k}_{ir} e_r = \frac{1}{2} \mathcal{J}^{-1} U_{rm} \left[\in_{mpk} U_{pl,i} U_{lk} + \in_{mlk} \left(U_{lp} U_{pk}\right)_{,i}\right] e_r \tag{44}$$

Condition of integrability of rotation can be found if we define: $\mathcal{K}_i = R^{-1} R_{,i} = \mathscr{k}_i \times I$ then from the condition of commutation: $R_{,ij} - R_{,ji} = 0$ one obtain (PIETRASZKIEWICZ, BADUR 1983b):

$$\mathcal{K}_{i,j} - \mathcal{K}_{j,i} + \mathcal{K}_i \mathcal{K}_j - \mathcal{K}_j \mathcal{K}_i = 0 \tag{45}$$

what Signorini (SIGNORINI 1943, p. 75) write with the use of the Darboux curvature vector as:

$$\frac{\partial}{\partial y_{r+1}} \mathscr{k}_r - \frac{\partial}{\partial y_r} \mathscr{k}_{r+1} - \mathscr{k}_r \times \mathscr{k}_{r+1} = 0, r = 1, 2, 3 \tag{46}$$

where y_r, $r = x, y, z$ means the Lagrangean coordinates. This result recalculated FERRARESE 1959. It can be written in an arbitrary curvilinear coordinate as ZHOUNG-HENG and DUBEY (1983), BADUR and PIETRASZKIEWICZ (1986) and BADUR (1993a):

$$\in^{ijk} \left(\mathscr{k}_{j,k} - \frac{1}{2} \mathscr{k}_j \times \mathscr{k}_k\right) = 0 \tag{47}$$

This formula does not appear in the Cosserats work (COSSERAT, COSSERAT 1896), although they want to show that the nonlinear conditions of insepa-rability of the deformations they obtained, after substituting the constrained rotations, lead to the six linear continuity conditions of Berré de Saint Venant.

Cosserats define the vector of linear rotation τ as $F = 1 + \varepsilon + \tau \times 1$ (COSSERAT, COSSERAT 1896, p. 135) and describe its derivative to be:

$$2\frac{\partial}{\partial y_i}\tau = \mathcal{k}_i + \tau(\mathcal{k}_i \otimes \tau) + \tau \times \mathcal{k}_i \tag{48}$$

After linearization of formulae (44) $\mathcal{k}_i = \mathcal{k}_{ir} e_r \approx \frac{1}{2} \in_{rlk} (\varepsilon_{lk})_{,i} e_r$ from eq. (47) we have:

$$\in^{ijk} \mathcal{k}_{j,k} = \frac{1}{2} \in^{ijk} \in_{rlm} (\varepsilon_{lm})_{,jk} e_r = 0 \tag{49}$$

There are six equations of Berré de Saint Venant which have been evaluated from constrained rotation approach. Yet other attempts to develop this line of reasoning one can find in papers by: ARIANO (1924), CESARO (1926), TONOLO (1930), STAZI (1976), FORTUNE and VALLEÉ (2001), where able to be shown more concise formulae which have an interpretation of the Bianchi identity.

Further developing of the moving frame concept within the Cartan differential forms calculus

It is a fact that the concept of the mobile reper has triggered thinking about new ways of "locating" the physical elementary object under reasoning within a time-space. Note, that the classical Newtonian mechanics is familiar only with "a simple placement" method. This brutal method leads to a concept of simple placement of a material particle within the Euclidean space. Usually, we call this method as "natural", but this method has nothing common with "the real nature". It is rather a result of anthropomorphic simplification only. Therefore, "the natural description" cannot be treated as only one possible in the whole continuum physics. Since, in relation to the classical natural description, that is originating in the Euclid geometry and the method of simple placement proposed by Newton, one can say that a new competitive way of placement based on the concept of "moving frame" has clearly appeared in the Cosserats doctrine. It forced a change in the mathematical tools of description because while the natural, metric, description uses mainly the "scalar product of vectors", the description of the mobile reper mainly uses "vector multiplication of vectors" and asymmetric components of tensors.

In the 1920s and 1930s, Élie Cartan, being against to Riemannian geometry, introduced calculus of differential forms, in which a new: "∧" multiplication played a major role; and instead of vectors and tensors, one-forms and two-forms have appeared. Cartan has radical changed the calculus and the form of governing equations (CARTAN 1935). For instance, the Cosserats continuity equations of deformation (see COSSERAT, COSSERAT 1896, eq. A) now take the form of the Mauer-Cartan structure equations (DELENS 1927): $d\theta + \omega \wedge \theta = 0$; $d\omega + \omega \wedge \omega = 0$. Unfortunately, the physical sense of these equations is the same, and exterior calculus change only the understanding of space-time Ricci metric. And inform as that the Riemannian or Euclidean metrics are too poor to be correct for exterior calculus framework. The biggest example of changes in the exterior calculus notation is the skew-symmetric field tension tensor, known in the Lie group theory as (CHAICHIAN, NELIPA 1984): $F_{\alpha\beta} = [D_\alpha, D_\beta] = -\partial_\alpha A_\beta + \partial_\beta A_\alpha + C^{\gamma\delta}_{\alpha\beta} A_\gamma$ – now it becomes a two--form (DE LEÓN et al. 2021) : $F = dA + A \wedge A$.

In general, the calculus based on differential forms is a good basis for "algebraization of physics" and for introducing the concept of "compensating potentials" into the differential geometry. One of the contemporary creators of the differential forms calculus and a continuator of Cartan's idea, Prof. J. Pommaret, believes that all mechanics should be rewritten in the language of differential forms in order to free oneself from the erroneous doctrine of "simple placement". In the literature there are many concepts of the new "location of an elementary object", for instance, nowadays p-brane plays a role of universal elementary object (POMMARIET 1989, 2014, BADUR 2009). EPSTEIN end DE LEON (1998) proposes to use the operation of embedding of an elementary object having $n-$ dimensions in a space-time continuum with $(n+m)$ −dimensions so as to build-up all the features of the elementary object with more sensitive time-space continuum features. As can be seen $(n+m)-$ dimensional space of Euclid, Riemann, projective or symplectic have to poor structure and should be abandoned as soon as possible (BADUR 2009, DE LEÓN et al. 2021).

Euclidean action and the corresponding group of transformations

For a long time before Cosserats, the principles of mechanics, whose role is preeminent in physics, have been the subject of discussions that touch upon their origin, nature, and implications. However, under the influence of the profound criticism the Lagrange variational approach to mechanics, the Cosserats interest in those studies has been renewed, and the aspects of the question have been modified from many points of view. It is not pointless to observe that if the founders of celestial mechanics believed very firmly that the study of discrete ensembles of points would reveal the secrets of nature then they did not perhaps have as much faith in Newton's laws of dynamics.

The Cosserats give a place the notion of symmetry group at the basis for mechanics. In his approach that notion has already played a significant role in the study of the principles of kinematics. For Cosserats, it seems that it must take on an importance in the discussion of the fundamentals of the physical sciences that is no less considerable because one can regard them an adequate translation of the idea of measurement due to the invariance that it implies.

In Cosserats case, whole of physical system is based on an energy Lagrangean W. Therefore, many principles of mechanics that are known without knowledge of W, must be accommodated to W as some restrictions. For instance, the d'Alembert principle can be introduced as an equivalent principle that relates only to the case where the action of deformation is completely separate from the kinetic action. The Cosserats assume the following principles:

– the equations of motion of the body with micro-rotation are the necessary condition of the stationarity of the action energy W;
– the action functional W is invariant with respect to the Euclidean group of transformation;
– the equation of motion has the same form in every inertial system i.e. they are invariant with respect to the Galilean transformation group (LANGE 1885).

The Cosserats introduced a notion of "transformation" not only in mechanics but also in physics. And they put a difference between the motion of coordinate system (passive transformation) and the motion of a material point within fixed coordinate system (active transformation). That last transformation is called a proper motion or active motion, or even displacement. Now, one sees immediately that the new transformation is again a motion. One expresses that by saying that when one composes the motions by performing them in succession, they will form a group that one calls the group of motions.

The Cosserats are first in the literature in introducing the condition of invariance under the group of Euclidean displacements which is a function of six parameters $a_1, a_2, a_3, \omega_1, \omega_2, \omega_3$:

$$\delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t\,,\delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t\,,\delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t \tag{50}$$

From condition of invariance the action $\delta W = 0$ with respect to the six-parameter Euclidean transformation group one can obtain the six conservation laws. It means, that, in principle, every model of continua contains m equations of motions and n accompanying conservation laws. Usually, from physics it follows that $m > n$. The discovery of the Euclidean group of symmetry was an important achievement, therefore Cosserats underline the invariance of action functional in the title of their next monograph *L'action euclidienne de déformation et de mouvement* (COSSERAT, COSSERAT 1909b). As we already mentioned, the unknown variables displacement and rotation are basic for describe variational laws of motion within the reference space. Thus, a change of the coordinate system in this space leads for a given configuration to a change of unknown variables for given localization in form of eq. (50). Making use of the relation eq. (50) from invariance condition one obtains:

$$\frac{\partial W}{\partial x} = 0, \frac{\partial W}{\partial y} = 0, \frac{\partial W}{\partial z} = 0 \tag{51}$$

$$\frac{\partial W}{\partial y_{,\varrho_b}} \frac{dz}{d\varrho_b} - \frac{\partial W}{\partial z_{,\varrho_b}} \frac{dy}{d\varrho_b} = 0, \frac{\partial W}{\partial z_{,\varrho_b}} \frac{dx}{d\varrho_b} - \frac{\partial W}{\partial x_{,\varrho_b}} \frac{dz}{d\varrho_b} = 0,$$

$$\frac{\partial W}{\partial x_{,\varrho_b}} \frac{dy}{d\varrho_b} - \frac{\partial W}{\partial y_{,\varrho_b}} \frac{dx}{d\varrho_b} = 0 \tag{52}$$

where the index $b = t, \varrho_1, \varrho_2, \varrho_3$. First three conditions (51) are nowadays interpreted to be the weak principle of momentum conservation. For the same reason the second three conditions (52) are called as the weak principle of moment of momentum conservation. Equations (51) and (52) are very important in the Cosserats monography, they are searched and developed for every Cosserats objects: beams, shells and 3D+time body. In the Cosserats (COSSERAT, COSSERAT 1909a) the form of (51) and (52) is quite the same and independent how many intrinsic coordinates was used.

Let us turn attention, that above conservation principles are valid only when the equations of motion for displacements and rotations are satisfied. This way of reasoning, but without any physical explanation has repeated nine years after by Emmy Noether (BADUR 2021). Everyone is astonished of the new, deeply anti-Newtonian, form of momentum conservation. But if we do not make the last condition but require the invariance of W no matter the equations of motion are satisfied or not, then we obtain so-called strong conservation principles – and fortunately, the strong conservation principle of momentum is fully equivalent to the second law of Newtonian motion.

Nowadays, the Euclidean group of symmetry, owing works of TOUPIN 1962, 1964, is assumed to be not six but seven parameters, such that the demand of invariance is equivalent to the existence of seven conservation laws for energy, momentum, and rotational momentum. If one takes into account possibility of the domain variables variations, then yet a set of additional conservation laws follows from the Euclidean group of symmetry – these are:

the conservation of pseudo-energy (BADUR 2021), the conservation of pseudo--momentum and conservation of pseudo-moment of momentum (KLUGE 1969, LAZAR, HEHL 2010).

Multidimensional branes as elementary statement of matter

The four-time unification concept, proposed by E. and F. Cosserats more than one hundred years ago, now is intensively developed in quantum physics within the framework of the brane field theory. It is surprising that the superstring theories are not just theories of one-dimensional objects. There are higher dimensional objects with dimensions from zero (points) to nine – such objects are called p-branes (MEISSNER 2013). In terms of branes, what we usually call a membrane would be a two-brane, a string is called a one-brane and a point is called a zero-brane. It is easily to find a precise analogy with the Cosserats mechanical-branes if we remember that string lives in ten-time dimensions, which means one real time dimension plus nine time-like dimensions. A special class of p-branes in string theory are called *D*-branes. Roughly speaking, a *D*-brane is a *p*-brane where the ends of open strings are localized on the brane. Now we can say that a Cosserats dream about a common mathematical theory for material continua and for electromagnetic and gravitational fields is fulfilled and preferred by Cosserats the differential geometry of fiber bounds plays a role of deductive leader (EPSTEIN, DE LEÓN 1998, LAZAR, HEHL 2010, DE LEÓN et al. 2021).

Four-time operators

The concept of gradient of deformation is a special relation of material change of immersed continua observed from the space-time point of view. For a point or a rigid body the deformation gradient is to be zero, since no any change inside that body can be observed. We can observe the "gradient of deformation" for a deformable line is a continuous set equipped with a finite small cross section area. If s is convective coordinates of curved material line than its deformation gradient is: $F_s = \mathrm{Grad}_s x = \partial_s x \otimes A^s = a_s \otimes A^s = F_{si} e_i \otimes A^s$. By the Hess analogy we may take the time gradient as $F_t = \mathrm{Grad}_t x = \partial_t x \otimes A^t = a_t \otimes A^t = F_{ti} e_i \otimes A^t$ where F_{ti} is interpreted as the velocity. Two-dimensional gradient on surface parametrized by ϱ_α is $F_2 = \mathrm{Grad}_2 x = \partial_\alpha x \otimes A^\alpha = a_\alpha \otimes A^\alpha = F_{\alpha i} e_i \otimes A^\alpha$, where $\alpha = 1, 2$, $i = x, y, z$.

In other words, a motion of single point is the time-parameter of space trace ($\varrho_0 = t$), a deformation of line is one parameter motion of trihedron:

$\varrho_1 = s$, a deformable surface with a set of two parameters $\varrho_\alpha, \alpha = 1, 2$ and a deformable 3D medium is governed by three parameters ϱ_i $i = 1, 2, 3$. In the presence of motion, one must add the time $\varrho_0 = t$ to these three geometric parameters ϱ_b, $b = t, 1, 2, 3$. The mathematical continuity that we assume in such a definition, leaves untouched at each point the trace of an invariable (i.e. rigid) solid; therefore, we can foresee that from a mechanical viewpoint moments will appear that are well known and are studied, since Euler and Bernoulli, along elastic flexible lines and on surfaces, and that Love and Helmholtz have tried to embed in a three-dimensional space.

This method is based on a simple extension of the natural convective, curvilinear coordinate system ϱ^i $i = 1, 2, 3$ having a time t, to a common system called "the four-time". Then the referential base system should be defined as: A^t, A^i – for three-dimensional body, $A^t, A^\alpha, \alpha = 1, 2$ – for dynamics of two-dimensional body (surfaces), A^t, A^s – for dynamics of one-dimensional body (rods), and A^t – for dynamics of rigid body. Let a Lagrangean gradient operator be defined as:

$$\mathrm{Grad}_4(\cdot) = (\cdot) \otimes \left[\frac{d}{dt} A^t + \frac{d}{d\varrho^i} A^i + \frac{d}{d\varrho^\alpha} A^\alpha + \frac{d}{d\varrho^s} A^s\right] \tag{53}$$

The intrinsic placement usually is described in terms of Lie algebra base adequate to the intrinsic group. Let us, for the von Helmholtz monodromy group (eq. 1) take for simplicity the local Cosserat reper base: d'_t, d'_1, d'_2, d'_3. In time $t = 0$ this reper takes the referential position: $d_t, d_1, d_2, d_3 = d_t, d_a$, $a = 1, 2, 3$. Since both reference vectors: A^t, A^i and d_t, d_a are Lagrangean-like and are a priori known, it is possible to define a shifter tensor $\mathrm{S} = S_{bi} d_b \otimes A^i$ which describes a connection between both systems. In particular case: $A^i = \delta^i_b d_b$. According to Cosserats, we do not loss any generality if we suppose that both referential systems coincides: $A^i = \delta^i_b d_b$ being the set of orthogonal vectors.

Two fundamental unknown fields of Cosserat continua are: the placement $x = x_i d_i = x'_i d'_i$ and the proper rotation: $R = d'_i \otimes d_i$, both measured from the intrinsic observer view point. Now, let us define the fundamental for the Cosserat four-time intrinsic formulation two basic measures of continuum velocities – both are expressed within the terms of referential gradient:

$$F = \mathrm{Grad}_4 x, F^* = \mathrm{Grad}_4 R \tag{54}$$

These gradients should be next "transported" into intrinsic frame where, according to the Cartan definition of a connection, they constitutes proper expression for intrinsic deformation measures, which Cosserats were called: the "geometrical velocities".

Cosserats measures of velocities and "geometrical velocities"

The first geometrical velocities, called by the name "translational geometrical velocities", taken collectively for zero-, one-, two- and three-dimensional bodies (the so-called the Cosserat p-brane) is defined to be:

$$\begin{aligned} \mathcal{V} = FR^{-1} &= [x\otimes(\partial_t A^t + \partial_s A^s + \partial_\alpha A^\alpha + \partial_i A^i)](d_j\otimes d_j') \\ &= (\partial_t x)\otimes d_t' + (\partial_s x)\otimes d_s' + (\partial_\alpha x)\otimes d_\alpha' + (\partial_i x)\otimes d_i' \\ &= x_{i,t}' d_i' + x_i'(\ell_t \times d_i')\otimes d_t' \text{ (rigid body)} \\ &= x_{i,s}' d_i' + x_i'(\ell_s \times d_i')\otimes d_s' \text{ (static of rods)} \\ &= x_{i,\alpha}' d_i' + x_i'(\ell_\alpha \times d_i')\otimes d_\alpha' \text{ (statics of surface)} \\ &= x_{i,b}' d_i' + x_i'(\ell_b \times d_i')\otimes d_b' \text{ (dynamics of 3D body)} \end{aligned} \tag{55}$$

Now, let recall the common notation of translational geometrical velocities, taken from Poisson's mechanics where: ξ,η,ζ- for a rigid body; ξ_s,η_s,ζ_s – for roods: $\xi_\alpha,\eta_\alpha,\zeta_\alpha$ – for surfaces; ξ_b,η_b,ζ_b – for 3D+time body. It means that eq.(54) can be shortly written as follows:

$$\mathcal{V} = FR^{-1}$$
$$= \begin{cases} \xi d_1'\otimes d_t' + \eta d_2'\otimes d_t' + \zeta d_3'\otimes d_t' = \nu_i d_i'\otimes d_t' \text{ (rigid body)} \\ \xi_s d_1'\otimes d_t' + \eta_s d_2'\otimes d_t' + \zeta_s d_3'\otimes d_t' = \nu_{is} d_i'\otimes d_s' \text{ (statics of rods)} \\ \xi_\alpha d_1'\otimes d_\alpha' + \eta_\alpha d_2'\otimes d_\alpha' + \zeta_\alpha d_3'\otimes d_\alpha' = \nu_{i\alpha} d_i'\otimes d_\alpha' \text{ (statics of surfaces)} \\ \xi_b d_1'\otimes d_b' + \eta_b d_2'\otimes d_b' + \zeta_b d_3'\otimes d_b' = \nu_{ib} d_i'\otimes d_b' \text{ (dynamics of 3D body)} \end{cases} \tag{56}$$

where we will be in accordance with the Poisson rigid body dynamics putting $d_t' = 1$. It is easy to find that $\mathcal{V} = FR^{-1}$ is a complicated, nonlinear function of the placement x and the rotation R. In particular, the Darboux vectors: $\ell_t, \ell_s, \ell_\alpha, \ell_i$, defined by the relations similar to the rigid body rotational motion: $d_{i,t}' = \ell_t \times d_i'$ according to Poisson's denotations are: $\ell_t = pd_1' + qd_2' + rd_3'$; $l_i = p_i d_1' + q_i d_2' + r_i d_3'$. Same times in the rigid body dynamics, the angular velocity vector:

$$\ell_t = -\tfrac{1}{2}(\dot{R}R^{-1})_\times \tag{57}$$

is called as the Euler-Poisson intrinsic vector. Since, in general we have the left (ℓ_b) and the right ($\mathscr{k}_b$) Darboux vectors ($b = t, s, \alpha, i$) and, generally, $\ell_b = R\mathscr{k}_b$, then for angular velocity ℓ_t there exist a right representation $\mathscr{k}_t$ such as: $\dot{R} = \ell_t \times R = R \times \mathscr{k}_t$. The left angular velocity ℓ_t is a space (intrinsic) measure and right angular velocity $\mathscr{k}_t$ is a body natural measure (FERRARESE

1959, CAPRIZ 2008). The Cosserats assumes that the quite similar "angular velocities" are related with spatial coordinates $\varrho_i, i = x, y, z$. Further, in this place one can see exactly, the "heard" of the Cosserats four-time continuum concept. Therefore, taking into account different denotations of the "rotational geometrical velocities" we can collect different results (rigid body, rods, shells, 3D) into one concise definition:

$$\mathcal{L} = -\frac{1}{2}(F^* R^{-1})_\times = \begin{cases} \ell_t \otimes d'_t = (p d'_1 + q d'_2 + r d'_3) \otimes d'_t = \ell_{it} d'_i \otimes d'_t \\ \ell_s \otimes d'_s = (p_s d'_1 + q_s d'_2 + r_s d'_3) \otimes d'_t = \ell_{is} d'_i \otimes d'_s \\ \ell_\alpha \otimes d'_\alpha = (p_\alpha d'_1 + q_\alpha d'_2 + r_\alpha d'_3) \otimes d'_t = \ell_{i\alpha} d'_i \otimes d'_\alpha \\ \ell_b \otimes d'_b = (p_b d'_1 + q_b d'_2 + r_b d'_3) \otimes d'_b = \ell_{ib} d'_i \otimes d'_b \end{cases} \tag{58}$$

Since, in the literature of Cosserats continuum, the tensor denotation dominate, we have write these components also in eq. (56) and (58) as: v_{ib} and ℓ_{ib}, $(b = t, s, \alpha, i)$. Both relations (56) and (58) can be easily written also in the Sudria vector notation: $\hbar_b, \ell_b$ as: $\mathcal{V} = \hbar_b \otimes d'_b$ and $\mathcal{L} = \ell_b \otimes d'_b$.

The main goal of the paper is not to destroy the continuum theory tradition where tensors are fundamental things and basic tools. We only try to explain the "secret line" of the Cosserats reasoning in which they are looking for better modeling of intrinsic properties of matter. Their mathematical concept is analogous with concept of a "monad" proposed by Leibnitz, the concept of "extensions" proposed by H. Grassmann, concept of "octonian" proposed by Caley, etc. (BADUR 2022). Mathematically, it is equivalent to the operation of taking the square on a scalar or vector. The best example of such successful operation is the Dirac equation of an electron written to be:

$$\text{Dirac spinor equation} = \sqrt{\text{Klein} - \text{Gordon equation}} \tag{59}$$

Trying to describe a continuum more precisely then the continuum governed by the Newtonian balance of the quality of motion, motivation of the Cosserats was to find a following system of equations:

$$\text{Cosserats equations of motion} = \sqrt{\text{Newtonian: } ma = f} \tag{60}$$

It is known fact from the gauge field theory (BASSET 1894) that the above "square operation" gives a possibility to introduce the intrinsic, local group of symmetry, which is hidden in the classical equation without "square". From this reason, the Cosserats continuum is described by "torsors" with six values in the Lie algebra. These six values are denoted by the Cosserats by specific order in their denotations, mainly, the repeated letters $\xi, \eta, \zeta, p, q, r$ which corresponds to six Lie algebra $\mathcal{T}_\alpha$ matrices (see eq. 4, 5 above).

The above formulae are one of the most marvelous in a whole classical field theory – it underline the main features of the four-time formulation of the local von Helmholtz group of symmetry. The Darboux vectors: $\ell_t, \ell_s, \ell_\alpha, \ell_i$ are the function of rotation only – it should be noted that, it is a single formulae: for a rigid body, rods, surfaces and 3D body – independently of which case is calculated.

But one can not to be impressed to resignation with the tensor description tradition. Since the Cosserat continuum is fully equivalently described within the natural Lagrangean and Eulerian descriptions. Owing to a numerous authors: HELLINGER (1914), SIGNORINI (1943), DE BORST (1991), LACHNER et al. (1994), LAKES (1995), SAWCZUK (1967), MALCOLM and GLOCKNER (1972), SHIELD (1973), SIMON and DELL'ISOLA (2017, 2018a, 2018b), a satisfied formulation in the language of tensors are available and are important in experimental foundations of the constitutive equations.

Intrinsic symmetry flux

Cosserats brothers have also introduced a concise system of internal measures of momentum and angular momentum fluxes. Independently of the dimension of body (a rigid body, rods, surfaces, 3D body) they proposed to use the following measures: A'_b, B'_b, C'_b; $b = t, s, \alpha, i$ for translational fluxes of symmetry and: P'_b, Q'_b, R'_b for rotational fluxes of symmetry. The order: A', B', C', P', Q', R' correspond with six $\mathcal{T}_\alpha$ matrices of Lie algebra. These measures appear in every Cosserats' bodies (COSSERAT, COSSERAT 1909a), therefore, we propose to introduce a one, unified, definition – for the translational fluxes:

$$\mathcal{T} = J\mathcal{V}^{-1}\sigma = \begin{cases} (A'd'_1 + B'd'_2 + C'd'_3)\otimes d'_t = \mathcal{T}_{it} d'_i \otimes d'_t \text{ (rigid body)} \\ (A'_s d'_1 + B'_s d'_2 + C'_s d'_3)\otimes d'_t = \mathcal{T}_{is} d'_i \otimes d'_s \text{ (statics of rods)} \\ (A'_\alpha d'_1 + B'_\alpha d'_2 + C'_\alpha d'_3)\otimes d'_t = \mathcal{T}_{i\alpha} d'_i \otimes d'_\alpha \text{ (statics of surfaces)} \\ (A'_b d'_1 + B'_b d'_2 + C'_b d'_3)\otimes d'_t = \mathcal{T}_{is} d'_i \otimes d'_b \text{ (dynamics of 3D body)} \end{cases} \tag{61}$$

and for rotational fluxes:

$$\mathcal{M} = J\mathcal{V}^{-1}\mu = \begin{cases} (P'd'_1 + Q'd'_2 + R'd'_3)\otimes d'_t = \mathcal{M}_{it} d'_i \otimes d'_t \text{ (rigid body)} \\ (P'_s d'_1 + Q'_s d'_2 + R'_s d'_3)\otimes d'_t = \mathcal{M}_{is} d'_i \otimes d'_s \text{ statics of rods)} \\ (P'_\alpha d'_1 + Q'_\alpha d'_2 + R'_\alpha d'_3)\otimes d'_t = \mathcal{M}_{i\alpha} d'_i \otimes d'_\alpha \text{ (statics of surfaces)} \\ (P'_b d'_1 + Q'_b d'_2 + R'_b d'_3)\otimes d'_t = \mathcal{M}_{is} d'_i \otimes d'_b \text{ (dynamics of 3D body)} \end{cases} \tag{62}$$

In above, $J = \det F$, and the nonsymmetrical momentum flux $\sigma = \sigma^{kl} e_k \otimes e_l$, the angular momentum flux $\mu = \mu^{kl} e_k \otimes e_l$ which are usually used for momentum and angular momentum balances (see: eq. 16). The above definitions of symmetry fluxes are independent of dimension of Cosserats' continua and, in the literature, are related to as the intrinsic formulation. Generally, the translational measures: $\nu_{ib}, \mathcal{T}_{ib}$ as well as the rotational one: $\ell_{ib}, \mathcal{M}_{ib}, b = t, s, \alpha, i$, are well known within the dynamics of rigid body, rods, material surfaces and 3D body. Especially, well known is a similarity between equations of the rigid body dynamics and rods statics – in the literature it has a name of the "Kirchhoff analogy" or "kineto-static analogy" (HESS 1884, GREEN, LAWS 1966, REISSNER 1981, HODGES 1990).

Gauge flux conservation

Among the requirements to be imposed on the lagrangean functional W the most important is the property of invariance under the action of gauge transformation. Classical Maxwell electrodynamics is the best example of the Lagrangean, which should satisfy simultaneously and independently the invariance under the action of 6-parameter Lorentz group and second one is the 1-parameter local $U(1)$ group of gauge symmetry. From each invariance property it follows the so-called currents of symmetry which are: six equations of conservation of the energy-momentum tensor (the current of the Lorentz group) and one equation $\partial_\mu j^\mu = 0$.

Analogically in the original Cosserats continuum there also exist two kind of group. The first one is the Euclidean group with constant, coordinate independent six translational and rotational parameters. The second one is local von Helmholtz group $\mathcal{H}(x,t) = T(3) \lhd SO(3)$, which play a role of $U(1)$ in Maxwell electrodynamics or $SU(2)$ in the Yang-Mills non-Abelian strong interaction. If the set of the Cosserats measures is simply written as eq. (7); $\mathcal{A}_b = \xi_b \mathcal{T}_1 + \eta_b \mathcal{T}_2 + \zeta_b \mathcal{T}_3 + p_b \mathcal{T}_4 + q_b \mathcal{T}_5 + r_b \mathcal{T}_6$ then, if lagrangean action is a function of Cosserats velocity measures: $W(\mathcal{A}_b)$ than one can define the following current symmetry fluxes:

$$\mathcal{J}^b = \frac{\partial W}{\partial \mathcal{A}_b} = A'_b \mathcal{T}_1 + B'_b \mathcal{T}_2 + C'_b \mathcal{T}_3 + P'_b \mathcal{T}_4 + Q'_b \mathcal{T}_5 + R'_b \mathcal{T}_6 \qquad (63)$$

It is nothing else like another form of the Cosserats stress and couple stress defined previously in $\mathcal{T}, \mathcal{M}$ (eqs. 61, 62). In original approach there is no d'Alembert principle introduced, then the Cosserats defines the $W(\mathcal{A}_b)$ to be fully coupled between deformation action and kinetic action. It is also the reason that we called this assumption as "four-time continuum". In order to explain the unexpected form of spatial $\mathcal{J}^i$ and temporal $\mathcal{J}^t$ and their energetical partners: $\mathcal{A}_i$, $\mathcal{A}_t$ the Cosserats underlines the common structure of gauge potential variation: $\delta\mathbb{M} = (\delta'\mathbb{M})\mathbb{M}$; $\delta(\mathbb{M})^{-1} = -\mathbb{M}^{-1}(\delta'\mathbb{M})$ what leads to (BADUR 1990, 1993a):

$$\delta\mathcal{A}_b = [\,\mathbb{I}\,\partial_b - \mathcal{A}_b]\delta'\mathbb{M} = \mathcal{D}_b\,\delta'\mathbb{M} \tag{64}$$

where the invariant (intrinsic) variation denoted by Cosserats as δ' is here also applied. Thus, using eq. (63) to the variation of lagrangean $W(\mathcal{A}_b)$ one obtains:

$$\begin{aligned}\delta\mathcal{A} = \int_{t_1}^{t_2}\iiint_{\mathcal{B}} \delta W\ d\varrho_i\ dt &= \int_{t_1}^{t_2}\iiint_{\mathcal{B}} \mathcal{J}^b \circ \delta\mathcal{A}_b\ d\varrho_i\ dt \\ &= \int_{t_1}^{t_2}\iiint_{\mathcal{B}} \mathcal{J}^b \circ [\,\mathbb{I}\,\partial_b - \mathcal{A}_b]\delta'\mathbb{M}\ d\varrho_i\ dt \\ &= \int_{t_1}^{t_2}\iiint_{\mathcal{B}} [\,\mathbb{I}\,\partial_b - \mathcal{A}_b]\mathcal{J}^b \circ \delta'\mathbb{M}\ d\varrho_i\ dt\ + bc.\end{aligned}$$

From the above it follows the common set of governing equation of motion written by means of covariant derivative $\mathcal{D}_b$ $(b = t, s, \alpha, i)$:

$$\begin{cases} \mathcal{P} = -\mathcal{D}_t\mathcal{J}^t \text{ (rigid body motion)} \\ \mathcal{D}_s\mathcal{J}^s + \mathcal{P} = 0 \text{ (statics of rods)} \\ \mathcal{D}_\alpha\mathcal{J}^\alpha + \mathcal{P} = 0 \text{ (statics of surface)} \\ \mathcal{D}_t\mathcal{J}^t + \mathcal{P} = -\mathcal{D}_t\mathcal{J}^t \text{ (dynamics of 3D body)} \end{cases} \tag{65}$$

Here, intrinsic sources of action, denoted by $\mathcal{P}$ are also written within Lie algebra base: $\mathcal{P} = \xi_0\mathcal{T}_1 + \eta_0\mathcal{T}_2 + \zeta_0\mathcal{T}_3 + p_0\mathcal{T}_4 + q_0\mathcal{T}_5 + r_0\mathcal{T}_6$ where the Cosserats denotations are used. There are Cosserats equations presented in the particular chapters of their monography. We hope that our revalorization of the Cosserats mathematical model will be useful in better understanding the structure of governing equations and the concept of many-time unification. We underline, also that the Cosserats concept of many-time unification is in a complete opposition to the concept of many-dimensional unification which is used in efforts by Norström, Kaluza, Silberstein and others pioneers in the general relativity and relativistic cosmology (BADUR 2021).

Conclusions

The Cosserats monography (COSSERAT, COSSERAT 1909a) is commonly known, but every new reader meets a problem of interpretation old Cosserats results. Sometime even the main motivations of the Cosserats are unknown. Therefore, one still needs a new revalorization of the classical formulas, for instance, the measures of Cosserats deformation. We recall that in the original Cosserats' memoir as well as in others numerous works (even within the linear theory) the definition of deformation measures is given ex cathedra – without any explanation. Therefore, the question about the theoretical sources and physical meaning of those measures is still actual. In all of discussions presented in the literature, the measures are introduced by more or less complete definitions, sometime being far from the original Cosserats one. In most of the works, additionally, the tensorial character of the Cosserats measures has been postulated. As a result, a new light on this problem is presented in our paper. Our revalorization of the Cosserats measures (see eq. 7) is unique and, we hope, final.

We have revalorized also the compatibility equations which are proposed in the early Cosserats work (COSSERAT, COSSERAT 1896). The new view on spatial--temporal compatibility is required if, one asserts possibility of a sort of "inertia anisotropy".

Yet another historical comment has been presented. Especially we are focused on a "intrinsic" approach in the moving frame. This approach is very important within the gauge theory of interactions; therefore we have entitled our paper: From solid mechanics backgrounds to modern field theory.

Let we come back to what we mentioned at the beginning of this paper. From the point of view of pure mathematics, the theory of "gauging continua" forms an interesting chapter of geometry of fiber bundle. From the point of view of the applications the value of this concepts is that they are helpful in studying the more difficult and general problems of mechanics, just as vector calculus has been for many years, and as tensor calculus is indispensable in the general relativity. Moreover, an appropriate calculation often enables us to grasp belter the essential features of the problem. For instance, the week and strong interactions cannot be nowadays described without the gauge potentials and symmetry groups. We should be glad if you got the impression that in this classical field of mechanics geometry there are still interesting questions, and that the study of these questions may be worthwhile for pure as well as the applied field theory.

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