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Diffusion, viscosity, convection and the boundary layer

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Abstract

We analyze the formation of the boundary layer and point to its affinity for diffusion. Based on Prandl's work, we discuss the properties of the boundary layer and its transition to the vortex state. We also describe the ways to reduce the unfavorable effects of the boundary layer encountered in nature.

Keywords: Viscosity, parabolicity, laminar flow, capillary flow, Albert Einstein's formula.

1. Introduction

Liquids, except for droplets held in their volume by surface tension forces, must be contained in some kind of vessel. Liquid flows also generally occur in vessels, such as channels, pipes, or tanks. Therefore, even potential flows, which according to theory do not depend on viscosity, are accompanied by boundary layer flows that are dependent on this phenomenon [1].

The equations of fluid motion, the so-called Navier-Stokes equations, contain a first time derivative and a second space derivative, as do the equations of diffusion and heat flow. They are all the parabolic equations. It turns out that the diffusion of particle velocities also manifests itself as the classical diffusion of particle position shifts.

The phenomenon of viscosity is related to the phenomena of diffusion and heat flow, and shares with them a dissipative nature. This is clearly seen in the kinetic theory of gases, where the heat conductivity and viscosity coefficients are expressed by the diffusion coefficient.

Namely, we have the diffusion coefficient $D = lu/3$, the thermal conductivity coefficient $\lambda = D\, f\, \mathrm{k_B}/2$, and the viscosity coefficient $\eta = D\, n\, m$. Here, l denotes the mean free path of the gas particle, u - its mean velocity, f - the number of degrees of freedom of the particle, n - the number concentration of gas particles (the number of particles in a unit volume), $m.$ - the mass of the particle, and $\mathrm{k_B}$ - the Boltzmann constant [2].

After neglecting the terms proportional to the viscosity coefficient η, the equations of a viscous fluid (Navier-Stokes) pass into the equations of an ideal fluid (Euler). The Euler equation is a first-order equation, the Navier-Stokes equation a second-order equation. The boundary condition requiring the vanishing of the normal velocity component at the stationary wall, $v_n = 0$, sufficient to uniquely solve the Euler equation cannot be sufficient to solve the equation requiring all components of the velocity vector to vanish at the wall, $\boldsymbol{v} = \boldsymbol{0}$.

On the other hand, the motion of the liquid particles whose trajectories lie farther from the walls differs little from the motion of a non-viscous liquid, and differs little from the potential motion. The potential motion of an inviscid liquid is, however, unavoidable to give the Euler equation, the integral of which cannot satisfy the boundary condition of a viscous liquid,

The main action of viscous forces, i.e. internal friction forces, occurs near the walls and is transferred to further portions of the liquid by means of particles that flow past the walls. Therefore, there is no theory of these movements yet. An approximate theory was given by Ludwig Prandtl in 1904, [3,4].

The existence of viscosity leads to the dissipation of energy, which ultimately turns into heat. Changes in flow velocity in the boundary layer are accompanied by a change in fluid temperature to a value equal to the wall surface temperature. If the Prandtl number (the ratio of kinematic viscosity to thermal diffusivity) is of the order of unity, then the order of magnitude of the boundary layer thickness δ at which both velocity and temperature change occur is inversely proportional to the square root of the Reynolds number $\sqrt{\mathcal{R}}$, [1].

Contemporary studies of the boundary layer distinguish three layers, of which only the first one closest to the wall is the classical Prandtl boundary layer [5]. Because it is in this layer that dissipative phenomena play a major role, in this chapter we limit our discussion to this layer.

Below we draw attention to the fact that the phenomena occurring on the margin of the movement of large masses of fluid in the boundary layer and the flows in capillaries, unless we are in a turbulent regime, are typical for diffusion.

The flow equations for viscous fluids are parabolic equations, as are the diffusion and heat flow equations. L. Prandtl pointed out that the substitution typical of diffusion phenomena reduces the two-dimensional flow equation to a one-dimensional equation. J.D. Cole's well-known observation is that the solution of the wave equation and viscosity equation can be expressed in terms of solutions of the diffusion equation. We also note that the development of the boundary layer and the filling of the capillary with a viscous fluid are typical of Brownian diffusion and are described by equations analogous to Albert Einstein's formula (1905). We analyze the formation of the boundary layer and point to its affinity for

2. Basic relations

The five quantities, three components of velocity $\boldsymbol{v}$, pressure p and density ρ fully determine the state of a moving liquid [1]. All these quantities are, in general, functions of coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ and time t. The function $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity of the liquid at a given point $\boldsymbol{x}$ of space, what means that it refers to a specific point of space, and not to specific particles of the liquid moving in space. The same applies to the quantities p and ρ. During the flow of a liquid, energy dissipation processes occur, related to the existence of internal friction, i.e. viscosity. In a viscous incompressible liquid, which we will deal with in the following, the phenomenon of viscosity is described by one coefficient designated by η.

If the fluid can be considered incompressible, then div $\boldsymbol{v}$ = 0. The equation of motion for an incompressible fluid has the form

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_k \partial x_k} \quad (1)$$

known as the Navier-Stokes equation (Claude-Louis Navier, 1822, Sir George Gabriel Stokes, 1842**).** It is often written in the following manner

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_k \partial x_k} \quad (2)$$

where the ratio

$$= \frac{\eta}{\rho} \quad (3)$$

is called the kinematic viscosity, while ν η itself is known as the dynamic viscosity[1].

We note that while the derivatives of the velocity on the left side of the equation are first order, the derivative of the velocity on the right side is second order and can be treated as a correction. If we neglect the phenomenon of viscosity and put $\eta = 0$, we obtain the equation of an ideal fluid

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} \quad (4)$$

(Leonhard Euler, 1752).

2. 1. Mathematical notation

In the above notation, the summation convention is accepted: whenever a small Latin index is repeated in an expression, it means the sum over all values of this index, and therefore in three-dimensional space the sum from 1 to 3, e.g. Laplacian in Eq.(1) means

$$\frac{\partial^2 v_i}{\partial x_k \partial x_k} = \frac{\partial^2 v_i}{\partial x_1 \partial x_1} + \frac{\partial^2 v_i}{\partial x_2 \partial x_2} + \frac{\partial^2 v_i}{\partial x_3 \partial x_3} \quad (5)$$

Similarly, the incompressibility condition can be written in the form

$$\operatorname{div} \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_k}{\partial x_k} = 0 \tag{6}$$

Equation (1) is the equation of motion expressed in terms of velocities

$$\rho \frac{\mathrm{d}\, v_i}{\mathrm{d}\, t} = \frac{\partial \sigma_{ik}}{\partial x_k} \tag{7}$$

The derivative on the left is called the substantial derivative and consists of two parts, as follows:

$$\frac{\mathrm{d}\, v_i}{\mathrm{d}\, t} = \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \tag{8}$$

The first part means the change during $\mathrm{d}\, t$ of the velocity at a point fixed in space. The second part is the difference between the velocities at the same instant at two points $\mathrm{d}\, \boldsymbol{x} = (\mathrm{d}\, x_1, \mathrm{d}\, x_2, \mathrm{d}\, x_3)$ apart, where $\mathrm{d}\, \boldsymbol{x}$ is the distance moved by the given fluid particle during the time $\mathrm{d}\, t$.

The tensor σ_{ik} on the right side of expression (7) denotes the stress tensor and for an incompressible medium it is given by the formula

$$\sigma_{ik} = -p\delta_{ik} + \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \tag{9}$$

The symbol δ_{ik} , known as Kronecker/s delta is defined as

$$\delta_{ik} = \begin{matrix} 1 & \text{if} & i = k \\ 0 & \text{if} & i \neq k \end{matrix} \tag{10}$$

We will still need a completely antisymmetric symbol, defined in three-dimensional space as follows

$$\epsilon_{abc} = \begin{matrix} 1 & \text{if } a, b, c \text{ are different and are an even permutation of 1,2,3} \\ -1 & \text{if } a, b, c \text{ are different and are an odd permutation of 1,2,3} \\ 0 & \text{iif two indices are equal} \end{matrix} \tag{11}$$

The important rule of calculation is the formula for the contracted product of two epsilons

$$\epsilon_{abk}\, \epsilon_{klm} = \delta_{al}\, \delta_{bm} - \delta_{am}\delta_{bl} \tag{12}$$

If necessary, for the sake of brevity we will denote the partial derivative with a comma in the spatial coordinate

$$\frac{\partial}{\partial x_k} (\ldots) = (\ldots)_{,k} \tag{13}$$

The ϵ_{klm} tensor can be used to write the cross product of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$

$$(\boldsymbol{A} \times \boldsymbol{B})_k = \epsilon_{klm}\, v_l\, v_m \tag{14}$$

and to write curl operations. For example, the *k*-th component of the curl operator on velocity $\boldsymbol{v}$ looks like this

$$(\operatorname{curl} \boldsymbol{v})_k = \epsilon_{klm}\, v_{m,l} \tag{15}$$

We further find that

$$(\boldsymbol{v} \times \operatorname{curl} \boldsymbol{v})_k = \epsilon_{kab}\, v_a \left(\epsilon_{bpq}\, v_{q,p}\right) = v_q\, v_{q,k} - v_p\, v_{k,p} \tag{16}$$

Hence

$$\frac{1}{2} (v_q v_q)_{,k} = v_p\, v_{k,p} + (\boldsymbol{v} \times \operatorname{curl} \boldsymbol{v})_k \tag{17}$$

The change in velocity $\mathrm{d}\boldsymbol{v}$ on the infinitesimal section $\mathrm{d}\boldsymbol{x}$ can be written as

$$\mathrm{d}v_i = \frac{\partial v_i}{\partial x_j} \mathrm{d}x_j \tag{18}$$

or

$$\mathrm{d}v_i = \left[\frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \right] \mathrm{d}x_k \tag{19}$$

The component

$$T_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{20}$$

is symmetric and represents the rate of change of the shape of the elementary volume of the liquid, the second component is antisymmetric and represents a rotation. According to (15) the pseudovector ω_k can be introduced

$$\omega_k = \frac{1}{2}\epsilon_{kij}\, v_{j,i} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{21}$$

which describes the velocity of infinitesimal rotations.

2.2. The principle of kinetic energy conservation

Helmholtz decomposition theorem states that certain differentiable vector fields can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector The kinetic energy of a unit volume of a liquid is

$$E = \frac{1}{2}\,\rho\, v_i v_i = \frac{1}{2}\,\rho\, v^2 \tag{22}$$

The change in energy is defined by the time derivative

$$\frac{\partial E}{\partial t} = \rho\, v_i\, \frac{\partial v_i}{\partial t} = \rho\, v_i\left(-v_k\,\frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} + \eta\,\frac{\partial^2 v_i}{\partial x_k \partial x_k}\right) \tag{23}$$

where we used the equation of motion (1).

Using the continuity equation (6) we write this relationship as follows

$$\frac{\partial E}{\partial t} = -\,\rho\,\frac{\partial}{\partial x_k}\left(\frac{1}{2}\, v_k v^2 + v_k p\right) + \eta\,\frac{\partial^2 v_i}{\partial x_k \partial x_k} \tag{24}$$

The quantity in brackets

$$\rho v_k\left(\frac{1}{2}\, v^2 + p\right) \qquad 25)$$

can be called the energy flux density vector, the last component in (22) containing the viscosity coefficient η represents the energy losses converted into heat.

2.3. Potential and rotational flow

Helmholtz decomposition theorem states that certain differentiable vector fields can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field (Hermann von Helmholtz, 1858)

$$v_i = v_i^{\text{pot}} + v_i^{\text{rot}} \tag{26}$$

We speak of potential flow when the velocity field $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ can be expressed by the gradient of a scalar field $\Phi(\boldsymbol{x}, t)$, called the potential

$$v_k^{\text{pot}} = \frac{\partial \Phi}{\partial x_k} \tag{27}$$

. The vector $\boldsymbol{v}^{\text{rot}}$, whose divergence vanishes identically, can always be represented as a rotation of a certain vector field. The vector $\boldsymbol{A}$ satisfying the equation

$$\epsilon_{kij}\, A_{j,i} = v_k^{\text{rot}} \tag{28}$$

is called the vector potential of the field $\boldsymbol{v}^{\text{rot}}$. It is always possible to choose the vector potential so that $A_{k,k} = 0$.

Since the divergence of the vector $\boldsymbol{v}^{\text{rot}}$ vanishes ($\epsilon_{kij}\, A_{j,ik} = 0$), the continuity equation (6) takes the form

$$\frac{\partial^2 \Phi}{\partial x_k \partial x_k} = 0 \tag{29}$$

and the function Φ is a harmonic function

We notice that by virtue of (27)

$$v_k^{\text{pot}}\,\frac{\partial v_i^{\text{pot}}}{\partial x_k} = \frac{1}{2}\,\frac{\partial}{\partial x_i}\left(\frac{\partial \Phi}{\partial x_k}\,\frac{\partial \Phi}{\partial x_k}\right) = \frac{1}{2}\,\frac{\partial}{\partial x_i}\left(v_k^{\text{pot}}\, v_k^{\text{pot}}\right) \tag{30}$$

At the same time, pursuant to (27)

$$\frac{\partial^2 v_i^{\text{pot}}}{\partial x_k \partial x_k} = \frac{\partial^2}{\partial x_k \partial x_k} \frac{\partial \Phi}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{\partial^2 \Phi}{\partial x_k \partial x_k} = 0 \qquad (31)$$

Next insert the division (26) into the Navier-Stokes equation (1) and obtain

$$\frac{\partial}{\partial t}\left(v_i^{\text{pot}} + v_i^{\text{rot}}\right) + \left(v_k^{\text{pot}} + v_k^{\text{rot}}\right)\frac{\partial}{\partial x_k}\left(v_i^{\text{pot}} + v_i^{\text{rot}}\right) = = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\eta}{\rho}\frac{\partial^2 v_i^{\text{rot}}}{\partial x_k \partial x_k} \qquad (32)$$

and we took advantage of the relationship (31). It can be seen that the potential and rotational flows are related to each other. By raising the order of the equation by applying the curl operator to both sides of the last equation we can isolate the rotational flow, and obtain

Writing down the Navier-Stokes equation (1) for v^{pot} , after using (28) and (29) we get

$$\rho\frac{\partial v_i^{\text{pot}}}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho\,\frac{1}{2} v_k^{\text{pot}}\, v_k^{\text{pot}} + p\right) = 0 \qquad (33)$$

If the velocity does not depend on time we get Bernoulli's equation

$$\rho\,\frac{1}{2} v_k^{\text{pot}}\, v_k^{\text{pot}} + p = \text{const} \qquad (34)$$

Thus, an increase in the speed of the flow occurs simultaneously with a decrease of pressure (Daniel Bernoulli, Hydrodynamica, 1738).

The potential flow equation (33) is identical to the Euler equation for an ideal fluid (4). Every potential flow satisfies the ideal fluid equation, but not every flow that satisfies the Euler equation is a potential flow.

From Eq.(17) we have

$$v_p\ \ v_{k,p} = \frac{1}{2}\left(v_q v_q\right)_{,k} - \left(\boldsymbol{v}\ \times \text{curl}\ \boldsymbol{v}\ \right)_k \qquad (35)$$

Applying the curl operator to both sides of Eq.(1) and using the last equality, we get

$$\frac{\partial}{\partial t}\ \text{curl}\ \boldsymbol{v} - \text{curl}\left(\boldsymbol{v} \times \text{curl}\ \boldsymbol{v}\right) = \frac{\eta}{\rho}\ \Delta\ \text{curl}\ \boldsymbol{v} \qquad (36)$$

This equation is of third order and contains only $\boldsymbol{v}$.

Example 1. Consider a velocity field whose absolute value in a cylindrical frame of reference is described by the formula

$$v = \frac{\alpha}{r} \qquad (37)$$

where α is a constant

The components of this velocity in a Cartesian frame are

$$(v_1, v_2) = \frac{\alpha}{r^2}(x, y) \qquad (38)$$

and can be derived from the potential

$$\Phi = \ \alpha \ln r \qquad (39)$$

The flow (37) is therefore potential.

Example 2. Now consider a velocity field whose absolute value in a cylindrical frame is described by the formula

$$v = \omega\, r \qquad (40)$$

where ω is a constant

The components of this velocity in a Cartesian frame are

$$(v_1, v_2) = \ \omega\,(-y, x) \qquad (41)$$

Therefore, according to Eq.(21) the rotation of the velocity field is

$$\frac{1}{2}(\text{curl}\ \boldsymbol{v})_3 = \omega \qquad (42)$$

The flow (40) is therefore rotational.

2.4. Boundary conditions

The equation of motion must be supplemented with boundary conditions that should be satisfied on the walls that limit the liquid. For an ideal liquid, these mean that the liquid cannot penetrate the surface of the walls. This means that on the immobile surfaces of the walls, the normal component of the liquid velocity should be zero.

In reality, there is no ideal liquid. Every real liquid has a certain, even small, viscosity. This viscosity may not manifest itself in the whole space, but even a small one plays a role in a thin boundary layer. It is the properties of motion in a thin, so-called boundary layer that determine the selection of one of the infinite number of solutions to the equation of motion.

In this way, when flowing through narrow gaps or tubes the entire volume of the fluid is actually a boundary layer.

2.5. Diffusion

Consider the diffusion equation

$$\frac{\partial f}{\partial t} = D\,\frac{\partial^2 f}{\partial x^2} \tag{43}$$

The function f represents a probability distribution depending on the position x and time t, and D is the diffusion coefficient.

We are looking for a solution to Eq(43) in the form $f = f(\xi)$, where

$$\xi = \frac{x}{\sqrt{t}} \tag{44}$$

We have

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial t} = -\,\frac{1}{2t}\,\xi\,\frac{\partial f}{\partial \xi} \tag{45}$$

and

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{t}\,\frac{\partial^2 f}{\partial\,\xi^2} \tag{46}$$

Substituting into Eq.(43) we get an equation with only one independent variable

$$-\xi\,\frac{\mathrm{d}f}{\mathrm{d}\xi} = 2D\,\frac{\mathrm{d}^2 f}{\mathrm{d}\,\xi^2} \tag{47}$$

After another substitution

$$\frac{\mathrm{d}f}{\mathrm{d}\xi} = g(\xi) \tag{48}$$

We have instead of (47)

$$-\xi g = 2D\,\frac{\mathrm{d}\,g}{\mathrm{d}\,\xi} \tag{49}$$

Hence

$$g = C\,\mathrm{e}^{-\,\xi^2/(4D)} \tag{50}$$

which, taking into account Eq.(44), is a typical form of the diffusion process.

Finally, the distribution function $f(\mathrm{x},\mathrm{t})$ is of the form:

$$f(x,t) = \frac{1}{\sqrt{4\pi D t}}\,\mathrm{e}^{-x^2/4Dt} \tag{51}$$

The mean square distance that a diffusing particle moves in a given x direction is

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2\, f(x,t)\mathrm{d}\,x = 2Dt \tag{52}$$

This is Albert Einstein's formula derived initially for Brownian movement [6].

By the way, note that the dimension of the diffusion coefficient D is the same as the kinematic viscosity coefficient ν and equal to cm^2/s. The same is the dimension of the velocity potential Φ.

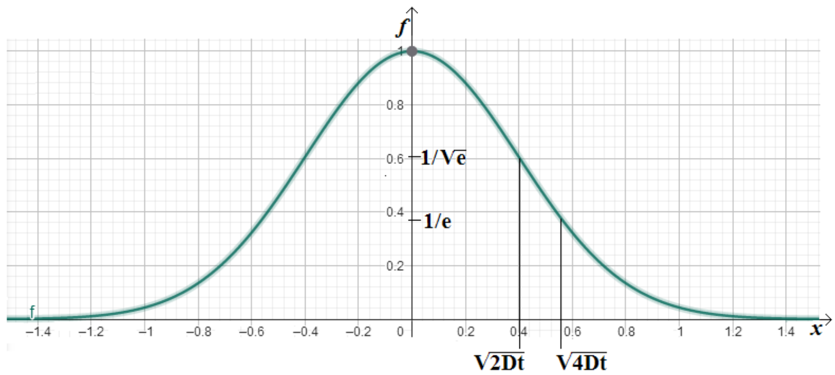

Figure 1. Distribution function f for $4Dt = 1/\pi$. For $x = \sqrt{2Dt}$ the function has a value of $1/\sqrt{\mathrm{e}} = 0.60653$, which is compared to a decrease to the value of 1/e

If the source is at a distance c from the absorbing wall located at $x = 0$, to solve the problem, we consider the auxiliary problem and introduce an ideal "image source". We have

$$f(x,0) = \delta(x-c) + \delta(x+c) \tag{53}$$

and the solution is, cf. Fig.2,

$$f(x,t) = \frac{1}{\sqrt{4\pi Dt}}\left[\mathrm{e}^{-(x-c)^2/4Dt} - \mathrm{e}^{-(x+c)^2/4Dt}\right] \tag{54}$$

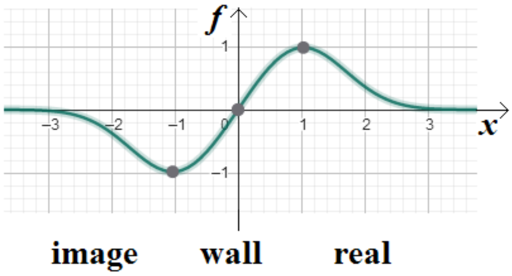

Figure 2. Distribution function f for initial value $f(x,0) = \delta(x-1)$ at a certain time t. The physical meaning has just a solution for $x \geq 0$

The Navier-Stokes equation (2), after removing the convective term and the potential gradient, becomes a diffusion equation with the diffusion coefficient ν. At the wall where the flow loses the velocity, its kinetic energy is converted into heat.

2.6. Parabolicity

Julian D. Cole considered the equation of a simple wave, namely he examined a special case of Eq.(1), in which the pressure gradient vanishes and the velocity has only one component $\boldsymbol{v} = (u, 0.0)$, i.e.,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2} \tag{55}$$

where $u = u(x,t)$ and $\nu = \eta/\rho$ is the kinematic viscosity, cf. Eq. (3). The presence of the first derivative in t and the second derivative in x indicates that this is a parabolic equation, analogous to the heat and diffusion equations, cf.(43). However, unlike them, equation (55) contains a nonlinear term.

J.D. Cole pointed out that if the function $\psi = \psi(x,t)$ satisfies the diffusion equation,

$$\frac{\partial \psi}{\partial t} = \nu\,\frac{\partial^2 \psi}{\partial x^2} \tag{56}$$

then the function $u = u(x,t)$ satisfying equation (55) is expressed by the function $\psi = \psi(x,t)$ with the relation

$$u(x,t) = -2\nu\,\frac{\partial}{\partial x}\ln\psi \tag{57}$$

In fact, after inserting (57) into (55) we get

$$2\nu\frac{\partial}{\partial x}\left[\frac{1}{\psi}\left(\frac{\partial\psi}{\partial t} - \nu\,\frac{\partial^2\psi}{\partial x^2}\right)\right] = 0 \tag{58}$$

that is, with the substitution (57), equation (55) is satisfied as long as the function $\psi = \psi(x,t)$ satisfies the diffusion equation (56).

Integrating Eq.(57) with respect to x we get

$$\psi(x,t) = C(t)\cdot\exp\left(-\frac{1}{2\nu}\int_0^x u(\xi,t)\,\mathrm{d}\,\xi\right) \tag{59}$$

The initial values are simply related. If

$$u(x,0) = u_0(x) \tag{60}$$

then

$$\psi(x,0) = \psi_0(x) = C_0 \cdot \exp\left(-\frac{1}{2\nu}\int_0^x u_0(x)\,(\xi)\,\mathrm{d}\,\xi\right) \tag{61}$$

A solution of Eq.(56) suitable for an infinite domain ($-\infty < x < \infty$) is given by

$$\psi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\nu t}\right) \psi_0(\xi x)\,\mathrm{d}\,\xi \tag{62}$$

After integration by parts Eq.(57), we get

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\nu t}\right) \exp\left[-\frac{1}{2\nu}\int_0^{\xi} u_0(\eta)\,\mathrm{d}\,\eta\right] u_0(\xi)\,\mathrm{d}\,\xi}{\int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\nu t}\right) \exp\left[-\frac{1}{2\nu}\int_0^{\xi} u_0(\eta)\,\mathrm{d}\,\eta\right] \mathrm{d}\,\xi} \tag{63}$$

an expression for $u(x,t)$ in term of its initial value [7,8].

It can be seen that the velocity $u = u(x,t)$ is expressed by a function typical of the diffusion equation.

Now, unlike in Example 1, Eqs.38-39, only the first component is expressed by the potential, Eq.(57). According to the continuity equation (6), the velocity component parallel to the y axis is $(\partial u/\partial x)\,y$, where u is given by (63). Therefore, in keeping with the definition (21), the velocity field in this case is a rotational field.

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3. Boundary layer

3.1. Couette flow

Let us begin with the descriptopn of planar Couette flow (Maurice Couette, 1980)[9]. Isaac Newton first defined the problem known now as Couette flow in Proposition 51 of his *Philosophiæ Naturalis Principia Mathematica*, and in Corollary 2.

Let the fluid be contained between two parallel planes moving with constant velocity u_0 relative to each other. The plane xz is chosen on one of them, with the x-axis directed in the direction of velocity u_0. All quantities depend, of course, only on y. From the Navier-Stokes equation [1[, for steady-state motion, we have

$$\frac{\mathrm{d}p}{\mathrm{d}y} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0 \tag{64}$$

Hence, we find that the pressure is constant, while the velocity

$$u = \frac{y}{h}\,u_0 \tag{65}$$

Any fluid that adheres to the wall is subject to the Couette flow near the wall.

The velocity of infinitesimal rotation in the Couette flow, after (21) is

$$\omega_3 = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) = -\frac{u_0}{2h} \tag{66}$$

In the boundary layer, the flow velocity increases with distance from the vessel wall, and therefore there is always infinitesimal rotation in the flow.

3.2.Prandtl estimation

Consider now the motion of an incompressible viscous fluid characterized by a large Reynolds number

$$\mathcal{R} = \frac{\rho u l}{\eta} = \frac{u\,l}{\nu} \tag{67}$$

Of course, $\mathcal{R}$ will be large when u or l is large, or η is small. It would seem that in this case, we would obtain a good approximation of the motion through the Navier-Stokes equations if we neglect the terms proportional to η.

However, such a simplification of the equations of viscous fluids is unacceptable. After neglecting the terms proportional to η, the Navier-Stokes equations become Euler equations. The Euler equations are first-order equations, while the Navier-Stokes equations are second-order equations

The boundary condition $v_n = 0$ on stationary walls is sufficient to uniquely determine the velocity field satisfying Euler's equations, and the condition $\boldsymbol{v} = \mathbf{0}$ on stationary walls, which applies to viscous fluids, cannot be imposed on the solutions of these equations.

Potential motion for viscous fluids, however, is impossible. The equations of motion for viscous fluids then transform into Euler's equations, whose integrals cannot satisfy the boundary conditions for viscous fluids.

In summary, for large $\mathcal{R}$, the motion of a viscous fluid in a region where there are no particle paths passing close to stationary walls may differ very little from potential motion, but it cannot be identical to it.

The main action of viscous forces occurs near the walls and is transmitted to further parts of the fluid via the fluid particles flowing close to the walls. Ultimately, for large $\mathcal{R}$ values, we can treat the fluid motion in the region away from the walls as the motion of an inviscid fluid and limit ourselves to studying the motion of the viscous fluid in the region where the fluid particles flow close to the walls. An approximate theory of such motion was presented by Ludwig Prandtl in 1904.

To obtain the equations of motion in the boundary layer in the plane case, L. Prandtl considers the plane motion $\boldsymbol{v} = (u, v. 0)$ of an incompressible fluid described by the equations [3]

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\,\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{68}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\,\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{69}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{70}$$

and estimates the orders of magnitude of the individual terms of these equations [3].

Assuming that the boundary layer thickness δ is very small compared to the characteristic length l, and that the viscous forces in the boundary layer are of the same order as the inertial forces, one obtains:

Hence

$$\delta = \frac{l}{\sqrt{\mathcal{R}}} \tag{71}$$

where $\mathcal{R}$ is the Reynolds number, Eq. (67).

From the relationship (71), we see that for large $\mathcal{R}$, the thickness δ is small. For example, taking l = 100 cm, u = 100 cm/s, $\nu = \eta/\rho$ = 0.01 cm^2/s (water at 20°C), we calculate $\mathcal{R} = 10^6$ and δ = 1 mm.

3.3. Pohl boundary layer estimation

After these elementary observations we can proceed to the description of the boundary layer

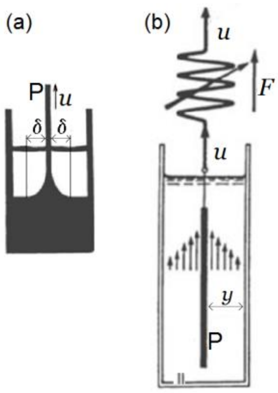

Figure 3. (a) When the plate P is pulled at a speed u, a boundary layer of thickness δ is formed on both sides of it. (b) The plate P is separated from each wall by a small distance y. The change in the fluid velocity is essentially linear over this distance

Robert Richard Pohl gave an interpretation of the phenomenon of boundary layer formation and a method of estimating the thickness of the boundary layer, assuming that it exists, cf. Fig3.

Let a liquid of density ρ have a cross-section δb in the boundary layer, where δ is the thickness of the layer and b is its width. Let the velocity of the liquid be u. In time t the liquid particles travel a distance $l = ut$, and the amount of liquid flowing through the cross-section hb is $m = \delta bvt$. This amount carries momentum $p = mu = \rho\delta bu^2 t$. The change in momentum, or the ratio p/t, should be equal to the force F, which maintains the fluid motion at speed v. This force results from shear stress and at the edge of the boundary layer this force is $= bl\,\eta\,u/\delta$. Comparing the change in momentum with the force gives

$$\rho\delta b u^2 = \eta\, bl\ \frac{u}{\delta} \tag{72}$$

and

$$\delta = \sqrt{\frac{\eta\, l}{\rho\, u}} \tag{73}$$

or

$$\delta = \sqrt{\frac{\eta}{\rho} t} \tag{74}$$

since $l = u\,t$. This is equivalent to relation (71).

It is seen that the boundary layer thickness δ is determined by the viscosity η and becomes zero with it. If the viscosity were zero, there would be no boundary layer at all. The width of the boundary layer given by formulae (71) or (74) is only conventional due to the dissipative nature of the layer, similarly to the width of the diffusion range, see Fig.1.

In his calculations, (Paul Richard) Heinrich Blasius (1908) [10] increased the coefficient for t in formula (74) fivefold, what, however, does not change the diffusive nature of the formula, cf. also [4].

3.4. Washburn propagation of liquid in Hagen-Poiseuille pipe

When flowing in a narrow tube, the boundary layer fills the entire liquid volume.

Suppose a liquid flows into a long capillary. Assume that it has already occupied the initial length l of the capillary. The sum p_S of atmospheric pressure and surface tension acts on the liquid's free side, while the pressure we will call p_H acts on the inflow side. The liquid movement is driven by the $p_H - p_S$ difference. Thus, the pressure difference tension $p_H - p_S$ moving the water in the capillary is constant. However, the pressure gradient decreases because the length l of the water column in the capillary increases. This is the idea behind Washburn's experiment [11-13].

According to the well-known formula, the volume of fluid $\mathrm{d}V/\mathrm{d}t$ flowing per unit time through a Hagen-Poiseuille tube of radius r under the influence of a pressure difference $p_H - p_S$ is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{p_H - p_S}{l} \cdot \frac{\pi r^4}{8\eta} \tag{75}$$

where $\boldsymbol{l}$ is the length of the fluid column in the tube, e.g. [1]. But

$$\mathrm{d}V = \pi r^3\, \mathrm{d}l \tag{76}$$

After inserting (76) into (75) and integrating, we get

$$l^2 = (p_H - p_S) \cdot \frac{r^2}{8\eta} \cdot t \tag{77}$$

where we have omitted the constant of integration.

In both equations (74) and (77), the growth of the boundary layer is proportional to the square root of time. This relation is similar to Albert Einstein's formula (52).

A similar relationship exists for the penetration of liquid into a porous medium.

3.3. Boundary layer description

After these elementary observations we can proceed to the description of the boundary layer, cf.Fig.4.

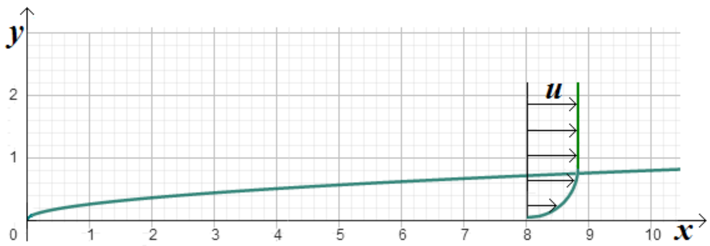

Figure 4. The conventional width of the boundary layer in an initial fluid flow stage along a flat plate increases proportionally to the square root of x, as shown by (73). The velocity u increases from 0 at the wall to the stationary Euler flow value

L. Prandl assumed that the viscosity of a liquid is small, so that it is only significant at the flow boundary, it is at the contact with the vessel wall. The viscosity is supposed to be so small that it can be neglected everywhere unless large differences in speed occur or an accumulating effect of viscosity takes place. This approach has proven fruitful, since it leads to mathematical formulations that make it possible to solve the problems, and the agreement with observation is satisfactory.

Estimating the terms of the equations allowed Prandtl to conclude that in the boundary layer the pressure does not change along the normal $\boldsymbol{n}$ to the contour,

$$\frac{\mathrm{d}\,p}{\mathrm{d}\,\boldsymbol{n}} = 0 \tag{78}$$

so that the pressure in the boundary layer can be assumed to be equal to the pressure of the inviscid fluid on the contour. The simplified (after neglecting small terms) equations (68-70) describe the motion in the boundary layer within the region, but only before the tangential velocity changes direction. These equations, in the vicinity of the point (0,0), if we assume the y-axis as the direction of the normal, are written as follows:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \nu\,\frac{\partial^2 u}{\partial y^2} \tag{79}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{80}$$

The pressure $p = p(x)$ should be taken as given in these relations.

Consider the one-dimensional steady flow equation. The velocity is therefore of the form . $\boldsymbol{v} = (u, 0, 0)$, where $u = u(y)$. For the sake of brevity, instead of (x_1, x_2) we write (x, y). With these reservations, Eq.(1) takes the form

$$\rho\,u\,\frac{\partial u}{\partial x} = -\,\frac{\partial p}{\partial x} + \eta\,\frac{\partial^2 u}{\partial y^2} \tag{81}$$

The first step of the numerical calculation is as follows. We assume that the velocity does not depend on x. Now consider the steady flow, Equations (79) and (80) take the form

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \nu\,\frac{\partial^2 u}{\partial y^2} \tag{82}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{83}$$

Suppose there is a function $\Psi = \Psi(x, y)$ such that

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\,\frac{\partial \Psi}{\partial x} \tag{84}$$

Introduce a dimensionless coordinate

$$\xi = \frac{y}{\sqrt{\nu x / a}} \tag{85}$$

where the constant a has the dimension of velocity.

Equation (81) resembles the diffusion equation (43) and substitution (85) is analogous to (44), except that instead of time t we have the coordinate x and instead of x we have y.

Suppose the function Ψ has the form

$$\Psi = \sqrt{a\,\nu x}\;\, g(\xi) \tag{86}$$

where the function $g(\xi)$ is to be determined. We substitute the velocity expressions (84) into equations (82) and (83). We have

$$\frac{\partial \xi}{\partial x} = -\frac{1}{2x}\,\xi \qquad \text{and} \qquad \frac{\partial \xi}{\partial y} = -\frac{1}{\sqrt{\nu x/a}} \tag{87}$$

Finally we get the nonlinear equation

$$\frac{\mathrm{d}^3 g}{\mathrm{d}\xi^3} + \frac{1}{2}\, g\, \frac{\mathrm{d}^2 g}{\mathrm{d}\xi^2} = 0 \tag{88}$$

Taking

$$g(\xi) = 2\, w(\xi) \tag{89}$$

instead of (88) we have

$$\frac{\mathrm{d}^3 w}{\mathrm{d}\xi^3} + w\, \frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} = 0 \tag{90}$$

After numerical solution, we obtain the dependence $u = u(\xi)$ shown in Fig.5.

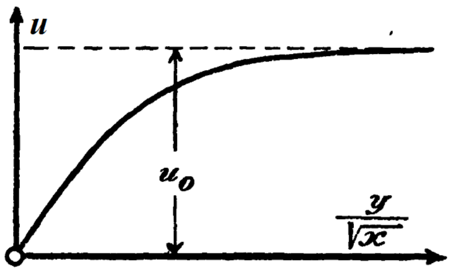

Figure 5. In boundary layer the fluid velocity increases from zero to bulk velocity u_0 , Prandtl [3]

According to formula (73), the boundary layer width increases with x, it is with the flow moving along the wall, Fig.5.

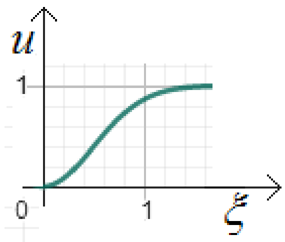

Figure 6. The function $u = u_0[1 - \exp\,(-\alpha\,\xi^2)\,]$ with u_0=1 and α=const, for comparison with the numerical solution shown in Fig.5

Meanwhile, the numerically found solution shown in Fig.5 resembles an exponential relationship

$$u = u_0\,[1 - \exp(-\alpha\,\xi^2)\,] \tag{94}$$

The graph of this function is shown for comparison in Fig.6.

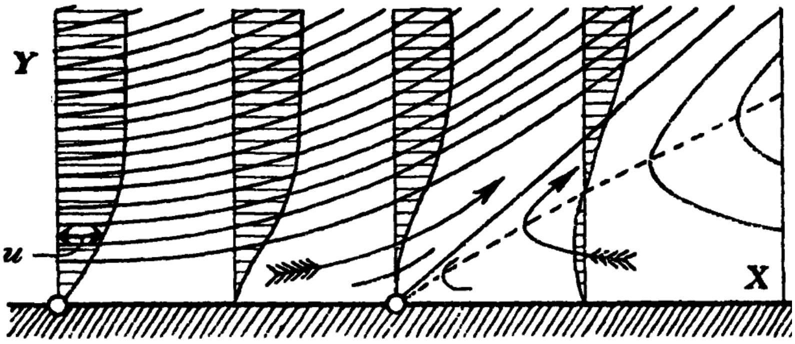

Figure 7. Development of the boundary layer, its separation from the wall and the formation of a vortex, Prandtl [3]

The most important result of Prandtl's investigations is that, in certain cases, the fluid stream detaches from the wall at a location determined by the external conditions (see Fig. 7).

Thus, a fluid layer, set in rotation by the viscosity against the wall, pushes itself out into the free fluid and, causing a complete transformation of the motion, plays the same role as in the Helmholtz separation layers.

As a closer discussion of Prandtl reveals, the necessary conditionfor the jet to detach is that there is an increase in pressure along the wall in the direction of the flow. The magnitude of this pressure increase in specific cases can only be determined from the numerical evaluation of the problem, which is still to be carried out. One plausible reason for the flow separation is that, when the pressure increases, the free fluid partially converts its kinetic energy into potential energy. However, the transition layers have lost a large part of their kinetic energy; they no longer possess enough energy to penetrate the region of higher pressure and will therefore move laterally away from it.

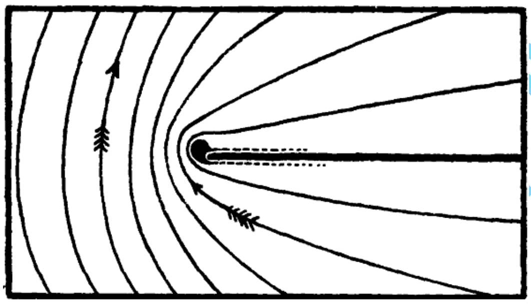

Figure 8. Initial vortex-free movement around plate wedge, Prandtl [3]

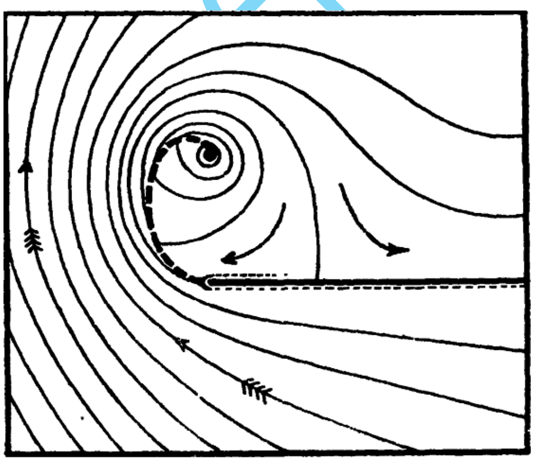

Figure 9. Spirally wound separation layer, Prandtl [3]

Figures 8 and 9 show the beginning of the movement around a wall projecting into the flow in two stages. The initial, vortex-free movement is rapidly transformed by a spirally wound separation layer (dashed line) emanating from the edge of the obstacle. The vortex moves further and further away, leaving behind stationary water the separation layer, which ultimately remains stationary.

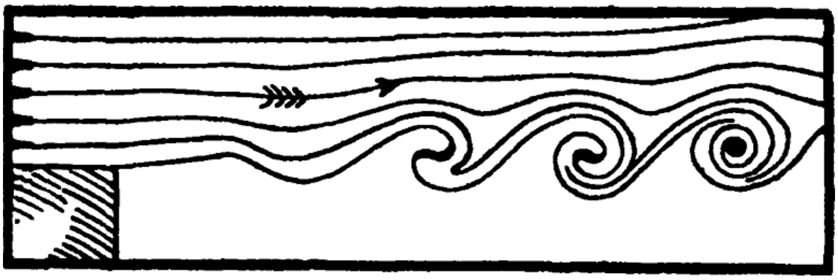

Figure 10. Vortex formation in a moving fluid flow (top) over a fluid at rest (bottom), Prandtl [3]

In Figure 10 one can see how this vortex coils up due to the intermeshing of the fluid flows. The lines in this figure are not streamlines, but rather those that would be obtained, for example, by adding colored liquid.

According to the foregoing, any treatment of a specific flow process can be divided into two interacting parts: on the one hand, there is the free fluid, which can be treated as frictionless according to Helmholtz's vortex laws; on the other, there are the transition layers at the solid boundaries, whose movement is regulated by the free fluid, but which, in turn, give the free movement its characteristic character by emitting vortex layers.

3.3. Boundary layer control

An overdeveloped boundary layer impedes the movement of floating objects, and generating vortices requires additional energy. Because the boundary layer increases with wall length, one of the basic methods is to shorten it and prevent unfavorable growth.

In nature, fish bodies are covered with scales. The scales are short enough that the boundary layer cannot expand too much, and the vortex that breaks off from the free end of the scale and falls into the cleft created by the next scale further facilitates swimming. Vortices behind the scales reduce drag thanks to the "rolling" effect. Specifically, a streamwise vortex is created behind each scale, which rolls along the surface. In that manner vortex replaces sliding reducing overall drag, [14-18].

Interwoven scales produce a streaky flow on the surface of fish as they swim, giving researchers a bio-inspired clue on how to reduce aerodynamic drag forces on aircraft. City, University of London Professor Christoph Bruecker and his team studied fish to determine how to increase aircraft speeds and reduce fuel consumption. Through their biomimetic study, Bruecker's team discovered that the fish scale array produces a zig-zag motion of fluid in overlapping regions of the surface of the fish, which causes periodic velocity modulation and a streaky flow that can eliminate Tollmien-Schlichting wave-induced transition to reduce skin friction drag by more than 25%, [19=20].

In engineering, boundary layer control refers to methods of controlling the behaviour of fluid flow boundary layers.

In technology there are also other methods of reducing the boundary layer, such as drilling holes in the wall and suction [21].

Conclusions

The flow equations for viscous fluids are parabolic equations, as are the diffusion and heat flow equations. L. Prandtl pointed out that the substitution typical of diffusion phenomena reduces the two-dimensional flow equation to a one-dimensional equation. J.D. Cole's well-known observation is that the solution of the wave equation and viscosity equation can be expressed in terms of solutions of the diffusion equation. We also note that the development of the boundary layer and the filling of the capillary with a viscous fluid are typical of Brownian diffusion and are described by equations analogous to Albert Einstein's formula (1905).

We examined the equations of fluid motion in a laminar boundary layer. Due to the thinness of the boundary layer, motion within it occurs essentially parallel to the flowing surface. The velocity perpendicular to the surface is small, which follows from the continuity equation. During the initial stage of its formation the thickness of the boundary layer is inversely proportional to the square root of the Reynolds number. This implies that the thickness of the boundary layer is described by a relationship analogous to Albert Einstein's formula. A similar relationship describes the depth of liquid penetration into the capillary as a function of time. These observations do not apply to a turbulent boundary layer

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