Technical Sciences, 2017, 20(4), 333–342

Quarterly peer-reviewed scientific journal
ISSN 1505-4675
e-ISSN 2083-4527
TECHNICAL SCIENCES
Homepage: www.uwm.edu.pl/techsci/

THE EFFECT OF INFLUENCE OF CONSERVATIVE AND TANGENTIAL AXIAL FORCES ON TRANSVERSE VIBRATIONS OF TAPERED VERTICAL COLUMNS

***Jerzy Jaroszewicz*[1], *Leszek Radziszewski*[2], *Łukasz Dragun*[1]**

[1] Faculty of Management
Bialystok University of Technology, Poland
[2] Department of Mechatronics and Machine Design
Kielce University of Technology, Poland

Received 15 December 2016, accepted 10 July 2017, available online 16 August 2017.

Key words: transverse vibrations, vertical cantilever, boundary value problem.

Abstract

The Cauchy function and characteristic series were applied to solve the boundary value problem of free transverse vibrations of vertically mounted, elastically supported tapered cantilever columns. The columns can be subjected to universal axial point loads which considerate – conservative and follower /tangential/ forces, and to distributed loads along the cantilever length. The general form of characteristic equation was obtained taking into account the shape of tapered cantilever for attached and elastically secured. Bernstein-Kieropian double and higher estimators of natural frequency and critical loads were calculated based on the first few coefficients of the characteristic series. Good agreement was obtained between the calculated natural frequency and the exact values available in the literature.

Introduction

Vertical building structures such as towers, chimneys and masts can be modelled using cantilever columns of variable cross sections, loaded at the free end with point-applied forces or along the axis with variable distributed loads. Cantilevers can be elastically supported to the base. Solutions to the boundary

Correspondence: Jerzy Jaroszewicz, Katedra Zarządzania Produkcją, Wydział Inżynierii Zarządzania, Politechnika Białostocka, ul. Ojca Tarasiuka 2, 16-001 Kleosin, e-mail: j.jaroszewicz@pb.edu.pl

value problem of critical loads and free transverse vibrations of a non-prismatic slender rod under the Euler buckling load have been reported in the literature (SZMIDLA, KLUBA 2011).

In paper (KUKLA, SKALMIERSKI 1993), the authors investigated an effect of axial loading on transverse vibrations of the Euler-Bernoulli beam of constant parameters. In JAROSZEWICZ, ZORYJ (2000), authors showed how easy it is to pass from the vibration boundary problem to critical load calculation in terms of divergence and flutter. The authors proposed an original solution to transverse vibration of the cantilever beam under the linearly variable load from dead load, which agreed with Euler's exact solution. In their analyses, the authors used the characteristic series method and introduced formulas for subsequent series coefficients using the influence function or the Cauchy function. To calculate basic natural frequency and critical forces, they used Bernstein-Kieropian double estimators, which helped find functional relationships between these values and the mass-elastic properties of the cantilever (JAROSZEWICZ, ZORYJ 1996). The influence function method in the analysis of the bending curve and relations of elastic supports of the beam with variable parameters was presented in JAROSZEWICZ et al. (2014). Such problems cannot be solved exactly for general function of variable cross section but in special cases, only when the equation is reduced to Euler's equation, special Bessel functions can be used to find the solution (ZORYJ 1982). The approach proposed by the author of this paper to apply the characteristic series method to the analysis of multi-parameter continuous systems seems warranted (JAROSZEWICZ, ZORYJ 1985, 1994. The literature reports analyses of this issue carried out using numerical and analytical methods including the MES, transfer matrix method and approximate methods based on energy principle such as those of Rayleigh-Ritz, Galerkin-Bubnow and Treffz (SOLECKI, SZYMKIEWICZ 1964).

Figure 1 shows three types of well known elastic rods loaded by non-conservative follower forces (BIDERMAN 1972). Figure 1*a* features a cantilever elastic column subjected to a follower torque M, whose vector follows the direction along the tangent to the deformed shaft axis. In Figure 1*b*, the cantilever rod has a rigidly fixed disc. Force P, maintaining the vertical orientation, does not connect to the material points of the disc but slides on its surface. Figure 1*c* shows the cantilever rod forced to the deformed axis of the rod. In all these cases, forces are external. To realize them, external follower devices should be used, such as aerodynamic propellers, pneumatic nozzles or similar systems as external energy sources. These problems are named after the researchers that were first to investigate them, Nikolai's problem, Reut's problem and the Beck's problem, respectively.

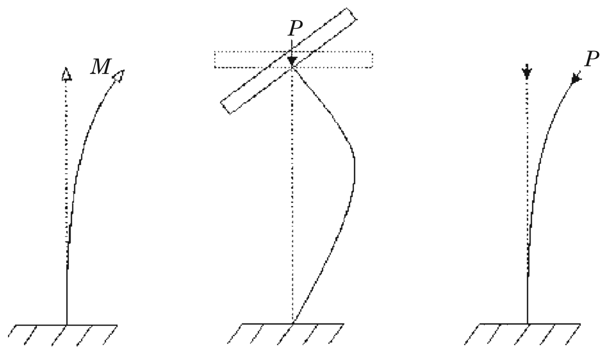

Fig. 1. Cantilever models under non-conservative forces
Source: BIDERMAN (1972).

In BIDERMAN (1972) the boundary problem of vibrations and critical loads is solved for vertical cantilevers elastically supported to the cone-shaped base.

The influence function and the partial discretization method were proposed in JAROSZEWICZ (1999) to solve the boundary value problem of free transverse vibrations of a non-homogeneous cantilever with a concentrated mass attached to its free end. In HAŠČUK, ZORYJ (1999), the authors showed that the influence function method can be effectively used to solve boundary value problems for one-parameter elastic systems with variable distribution of parameters. Universal form of a characteristic equation for a vertical cantilever, which does not

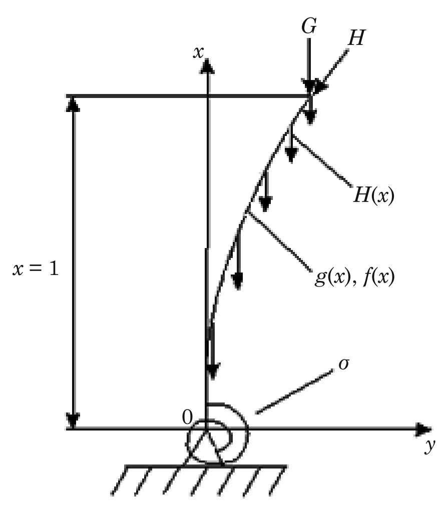

Fig. 2. The model of variable cross section column elastically secured bar with point loads: conservative G and tangential H and with distributed variable load $N(x)$

depend on the cantilever shape or the kind of axial load. The shape of the cantilever and the kind of axial load (JAROSZEWICZ, ZORYJ 1997) were taken into consideration in the form of the influence function. Form of influence function suitable to arbitrary change of a cantilever cross-section and distributed axial load were received.

In this paper, the effect of axial loads on the transverse vibrations of cantilevers with constant and variable cross sections is investigated. The cantilever model under investigation is shown in Figure 2. The following notation is used in Figure 2: $f(x)$, $g(x)$, $N(x)$ are the functions describing the distribution of the flexural rigidity, mass and axial load along the cantilever axis, G and H are the conservative and tangential forces acting at the free end of the cantilever, x and y are Cartesian coordinates, l is the length of the cantilever.

The study involved detailed investigations of vertical tapered cantilevers with geometry characterized by taper ratio γ and load parameter η which take into account conservative force G and tangential force H.

Solving the boundary problem of vibrations of a cone under conservative force *G* and tangential force *H*

The boundary problem reads

$$(f(x)y'')'' + py'' - \Omega^2 g(x)y = 0 \qquad 0 < \sigma < 1 \tag{1}$$

For the homogeneous or uniform cone, suitable mass-elastic parameters could be incorporated in these formulas

$$f(x) = (1 - \gamma x)^4, \quad g(x) = (1 - \gamma x)^2, \quad \gamma = \frac{1}{h_1}, \quad \Omega^2 = \alpha\omega^2, \quad p = \frac{l^2}{f_0}(G + H),$$

$$m_0 = g(x)\,|_x = 0$$

where

h – is length of cone which is parts of sharp cone which length is l,

$J(x)$ – moment of inertia cross section.

E is Young's modulus, I_0 denotes the moment of inertia of the cross section at the fixed end, m_0 is the unit mass corresponding to the cross section at the fixed end, p and ω are the load and frequency parameters, η is the parameter of non-conservatively, and γ is the taper ratio for conical cantilevers and σ is rigidity coefficient of elastic supports.

$$\alpha = \frac{m_0 l^4}{f_0} \qquad \eta = \frac{H}{G + H} \tag{2}$$

The boundary condition for $\gamma(x) \equiv 0$ is as follows

$$y(0) = 0, \quad y'(0) + \sigma''(0) = 0, \quad f(x)y''(x)|_{x=l} = 0, \quad ((f(x)y''(x))' + Gy'(x))|_{x=l} = 0 \tag{3}$$

The boundary conditions in the case when attached cantilever $\sigma = 0$ and with consideration for $N(x)$ can be written as

$$y(0) = y'(0) = 0, \quad f(x)y''(x)|_{x=1} = 0 \text{ and } G \equiv 0 \tag{4}$$

$$(f(x)y''(x)') - N(x)y'(x)|_{x=1} = 0 \tag{5}$$

As in ZORYJ (1982), the general solution has the form

$$y(x,\alpha) = K(x,\alpha) + \dot{K}(x,\alpha) + \ddot{K}(x,\alpha) + \dddot{K}(x,\alpha) \tag{6}$$

where:
$K(x,\alpha)$ – Cauchy's function derivatives with respect to $\dot{K}(x,\alpha) + \ddot{K}(x,\alpha) + \dddot{K}(x,\alpha)$.

Substituting expression (6) into conditions (3–4) yields the system of equations with respect to unknown constants C_0, C_1, C_2 and C_3. Equating the determinant of the above equation to zero, we obtain the characteristic equation.

$$\begin{aligned} \nabla \equiv f(x)[K'(x,\alpha)K'''(x,\alpha) - K'''(x,\alpha)K''(x,\alpha] + \\ +pN(x)[K'(x,\alpha)K''(x,\alpha) - K''(x,\alpha)K'(x,\alpha)] = 0|_{\substack{x=1 \\ \alpha=0}} \end{aligned} \tag{7}$$

It is common practice in engineering neglect some loads, namely, $N(x) \equiv 0$ and $G = H - 0$. In this case, characteristic equation (6 and 7) becomes as

$$\nabla \equiv [K''(x,\alpha)\dot{K}'''(x,\alpha) - K'''(x,\alpha)\dot{K}''(x,\alpha)] - K''(x,\alpha)\dot{K}(x,\alpha)]| = 0|_{x=1} \tag{8}$$

The above equations are a direct consequence of the definition of Cauchy's function (JAROSZEWICZ, ZORYJ 1997, 2014). Equation (8) is a universal characteristic equation, taking into account all considered cases of longitudinal load and any change in transverse cross section of the bracket. As will be shown in the following paragraphs, the basic problem of solving the equation (8) is to determine the appropriate form of Cauchy's influence function for the case in question.

The Cauchy function with respect to the four variables corresponding to the bracket of any continuous load ($N(x)$) has the following form JAROSZEWICZ, ZORYJ (1997):

$$K(x,\alpha,p,\mu) = f(\alpha)\sum_{i=0}^{\infty}\mu^i I_i(x,\alpha,p) \tag{9}$$

where:

$$I_i(x\alpha,p,\mu) = \int_{\alpha}^{x} g(t)I_0(x,t,p)I_{i=1}(x,\alpha,p)\mathrm{d}t,$$

$$I_0(x,\alpha,p) = \sum_{k=0}^{\infty}(-p)^k V_k(x,\alpha),$$

$$V_k(x,\alpha) = -\int_{\alpha}^{x} N(t)V_0(x,t)W'_{k-1}(t,\alpha)\mathrm{d}t,$$

$$V_0(x,\alpha) = \int_{\alpha}^{x}\frac{(x-s)(s-\alpha)}{f(s)}\,\mathrm{d}s.$$

The form of the influence function (9) ensures that the characteristic equations will be power series with respect to the parameter with the coefficients A_k dependent on the load parameter p:

$$\sum_{k=0}^{\infty} A_k(l,0)\mu^k = 0 \tag{10}$$

Cantilever loaded conservative and tracking forces

The coefficients of the characteristic series (10) in this case ($N(x)$ 0, $M = 0$) can be determined using the formulas (JAROSZEWICZ, ZORYJ 1997):

$$A_k(x,\alpha) = \sum V_{i,k-i}(x,\alpha) \text{ '4 } p(1-\eta)W_{i,k-i}(x,\alpha) \tag{11}$$

where

$$V_{i,j}(x,\alpha) = (J_i'' J_j''' - J_i''' J_j'')f^2(\alpha)f(x) \tag{12}$$

$$W_{i,j}(x,\alpha) = (J_i J_j'' - J_i'' J_j')f^2(\alpha) \tag{13}$$

The first three coefficients of the series defined by (11) are:

$$A_0(x,\alpha) = V_{00} - p(1-\eta)W_{00} \tag{14}$$

$$A_1(x,\alpha) = V_{01} + V_{10} - p(1-\eta)(W_{01} + W_{10}) \tag{15}$$

$$A_2(x,\alpha) = V_{02} + V_{11} + V_{20} - p(1-\eta)(W_{02} + W_{11} + W_{20}) \tag{16}$$

Considering the truncated cone support, for which stiffness and mass functions have been given the following form of function $J_i(x, \alpha)$ i $U(x, \alpha)$ (JAROSZEWICZ, ZORYJ 1997):

$$J_0(x,\alpha) = \int_\alpha^x \frac{(x-t)U(t,\alpha)}{f(t)}\mathrm{d}t \tag{17}$$

$$J_i(x,\alpha) = \int_\alpha^x g(t)J_0(x,t)J_{i-1}(t,\alpha)\mathrm{d}t \tag{18}$$

$$U(x,\alpha) = \frac{1}{\varphi(x,\alpha)}\sin[\varphi(x,\alpha)(x-\alpha)] \tag{19}$$

$$\varphi(x,\alpha) = \frac{\sqrt{p}}{(1-\gamma x)(1-\gamma\alpha)} \tag{20}$$

with the help of which factors were built (14), (15), (16).

The equating zero to the first coefficient of series (10), we obtain the equation, whose element with respect to the variable p gives the critical load in Euler's sense for the bracket:

$$\eta + (1-\eta)(1-\gamma)\left[\cos\frac{\sqrt{p}}{1-\gamma} + \frac{\gamma}{\sqrt{p}}\sin\frac{\sqrt{p}}{1-\gamma}\right] = 0 \tag{21}$$

Example of calculation double estimators of base frequency: lower ω_-^2 and higher ω_+^2 and critical loads

The case of the fixing rigidity $\delta = 0$ and non-conservative load with force $H(G = 0)$ is considered as first. Table 1 summarizes the natural frequencies of the rod depending on the compression force H. Table 2 summarizes the calculated results for the frequencies of the cantilever with the clamping elasticity $\delta \neq 0$ taken into account.

Table 1

Results of calculation natural frequency estimators for the attached cantilever $\delta = 0$

$\gamma = 0$			$\gamma = 0.2$			$\gamma = 0.5$			$\gamma = 0.7$		
p	ω_-^2	ω_+^2	p	ω_-^2	ω_+^2	p	ω_-^2	ω_+^2	p	ω_-^2	ω_+^2
0	12.36	12.36	0	14.81	14.82	1	26.54	26.66	1	43.98	44.60
1	13.25	13.26	1	16.44	16.46	2	33.24	33.50	2	71.27	74.79
5	17.77	17.79	5	25.46	25.55	3	42.06	42.68	3	104.83	119.60
10	26.70	26.81	10	48.81	50.04	4	53.94	55.53	3.1	108.56	125.60
15	43.32	44.10	11	57.05	59.50	5	70.35	75.18	3.2	112.54	132.80
19.5	78.01	96.40	12	67.72	73.38	6	94.36	126.80	–	–	–
19.6	79.32	101.98	13	82.20	104.20	–	–	–	–	–	–

Source: HAŠČUK, ZORYJ (1999).

Table 2

Calculation results from natural frequency estimators for the elastically secured cantilever $0<\delta<20$

$\gamma = 0$			$\gamma = 0.2$			$\gamma = 0.6$			$\gamma = 0.8$		
σ	ω_-^2	ω_+^2	σ	ω_-^2	ω_+^2	σ	ω_-^2	ω_+^2	σ	ω_-^2	ω_+^2
1	2.726	2.727	1	3.262	3.263	1	7.016	7.023	1	11.804	11.843
5	0.664	0.665	5	0.786	0.787	5	1.776	1.777	5	3.037	3.039
10	0.342	0.343	10	0.404	0.405	10	0.917	0.918	10	1.573	1.574
15	0.229	0.230	15	0.271	0.272	15	0.618	0.619	15	1.061	1.062
20	0.173	0.174	20	0.204	0.205	20	0.466	0.467	20	0.800	0.801

Source: HAŠČUK, ZORYJ (1999).

In the Figure 8 shows the results of the critical load calculation for the truncated cone obtained from equation (21) (JAROSZEWICZ, ZORYJ 1997).

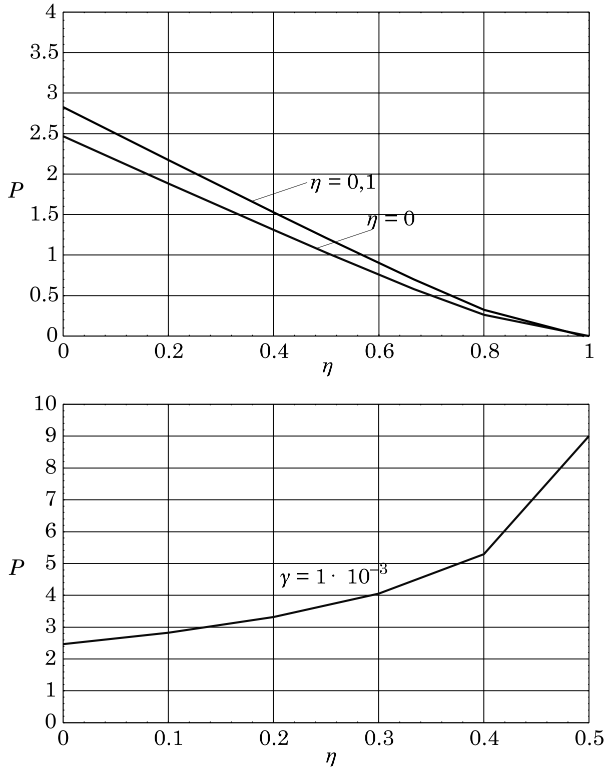

Fig. 3. Euler critical force for a cantilever cone under conservative and follower forces

Conclusion

In the paper shows that the influence function method can be an effective tool for solving the boundary problem of single- and two-parameter elastic systems with variable distribution of parameters. The universal characteristic equation (8), (10) for the vertical tapered cantilevers, which does not depend on the beam shape and axial load type, has been recorded. The shape and type of load is taken into account in the form of an influence function (9) that corresponds to any change in cross-section of the support and continuous axial load with the condition that functions describing stiffness, and the continuous load was total.

In detail, a conical shaped cone with a convergence coefficient was considered γ, γ, which is laden with the conservative force G, the tracking force H. The share of forces G and H is determined by the coefficient of conservatism η. In this case, the integral expressions for the first three members of the characteristic series (14), (15), (16) are derived. The general form of the kth member of the series (11) was also recorded.

Vertical tapered cantilevers with geometry characterized by coefficient γ and subjected to conservative force G and tangential force H defined by load parameter η were investigated in detail.

The method employing characteristic series and equal-tail estimators used in this paper allows obtaining functional relationships for natural frequency estimators and critical loads of flutter and divergence types, which in turn facilitates optimization of mass elastic parameters of the system for the reduction of dynamic loads – the loss of stability and for preventing resonance. This method can be of use in engineering calculations.

References

BIDERMAN V.L. 1972. *Prikladnaja teorija mechaničeskich kolebanij*. Vyssšaja Škola, Moskva.

HAŠČUK P., ZORYJ L.M. 1999. *Linijni modeli diskretno-neperervnyh mechanicznych system*. Lviv, Ukrainski technologii, 372.

JAROSZEWICZ J. 1999. *The effect of non-homogenous material properties on transverse vibration of elastic cantilever*. JAM, Kiev, 35(6): 103–110.

JAROSZEWICZ J., ZORYJ L. 1985. *Free transversal vibrations of a cantilever beam with variable cross section*. Eng. Trans., 33(4): 537–547.

JAROSZEWICZ J., ZORYJ L. 1994. *Transversal vibrations and stability of beams with variable parameters*. Int. Appl. Mech.-Eng. Tr., 30(9): 713–720.

JAROSZEWICZ J., ZORYJ L. 1996. *Critical Euler load for a cantilever tapered beam.* J. Theor. Appl. Mech., 4(34): 843–851.

JAROSZEWICZ J., ZORYJ L.M. 1997. *Metody analizy drgań i stateczności kontynualno-dyskretnych układów mechanicznych*. Politechnika Białostocka, Białystok.

JAROSZEWICZ J., ZORYJ L. 2000. *Investigation of axial loads on transverse vibrations of vertical cantilevers of variable parameters*. JAM, Kiev, 36(9): 1242–1251.

JAROSZEWICZ J., ŻUR K., DRAGUN Ł. 2014. *The influence function method in analysis of bending curve and relations of elastic supports of beam with variable parameters*. Journal of Theoretical and Applied Mechanics, 52(1): 247–255.

KUKLA S., SKALMIERSKI B. 1993. *The effect of Axial Load on Transverse Vibrations of an Euler Bernoulli Beam*. J. Theor. and Appl. Mech., 2(31).

SOLECKI R., SZYMKIEWICZ J. 1964. *Układy prętowe i powierzchniowe, obliczenia dynamiczne*. Arkady, Warszawa.

SZMIDLA J., KLUBA M. 2011. *Stateczność i drgania swobodne niepryzmatycznego układu smukłego poddanego obciążeniu eulerowskiemu*. Modelowanie Inżynierskie, 41: 385–394.

ZORYJ L.M. 1982. *Universal characteristic equations in problems on the vibrations and stability of elastic systems*. Tverd. Tela, 6: 155–162.