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HEURISTIC DERIVATION OF BRINKMAN'S SEEPAGE EQUATION

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Abstract

Brinkman's law is describing the seepage of viscous fluid through a porous medium and is more acurate than the classical Darcy's law. Namely, Brinkman's law permits to conform the flow through a porous medium to the free Stokes' flow. However, Brinkman's law, similarly as Schrödinger's equation was only devined. Fluid in its motion through a porous solid is interacting at every point with the walls of pores, but the interactions of the fluid particles inside pores are different than the interactions at the walls, and are described by Stokes' equation. Here, we arrive at Brinkman's law from Stokes' flow equation making use of successive iterations, in type of Born's approximation method, and using Darcy's law as a zero-th approximation.

Introduction

Many problems of interest, involve the motion of fluid through a porous solid, which interacts at every point with the diffusing fluid (MORSE, FESHBACH 1953). The classical equation describing the fluid seepage through a porous solid is known as Darcy's law

$$v = -\frac{K}{\eta}\nabla(p + U) \tag{1}$$

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It expresses the fluid velocity v (understood as the filter velocity rather than as the true velocity in pores) by the gradient of the pressure p and the volume force potential U. The permeability coefficient K describes the porosity of the solid, and η denotes the viscosity of the fluid.

Equation (1) was proposed by Henry Darcy in 1855 on the basis of his experiments (cf. DARCY 1856), and can be motivated by contemporary asymptotical methods (cf. SANCHEZ-PALENCIA 1980). The equation patterned on others transport equations (Fourier's, Ohm's) does not render aptly the specificity of the fluid. A basic objection is that any viscous shear tensor can be derived from it, as the viscous shearing has been neglected. Related to this objection are difficulties in posing the boundary conditions, for example for problems in which the fluid flows concomitantly through porous medium and adjoining empty space.

In the versatile physical heritage of Henri Coenraad Brinkman involving quantum physics, physical chemistry, applied physics (cf. e.g. BRINKMAN, KRAMERS 1930, BRINKMAN 1947) one finds also his equation describing seepage of the fluid through porous medium, cf. (BRINKMAN 1949). The equation gives the following expression for the fluid velocity

$$v = \frac{K}{\eta} [- \nabla(p + U) + \eta' \Delta v] \tag{2}$$

Equation (2) is known as Brinkman's equation or Brinkman's seepage law. It is known also as Darcy-Brinkman's equation (cf. VALDES-PARADA et al. 2007).

In comparison with Darcy's Equation (1) the term with Δv was added at the right hand side of Equation (2). The coefficient η' (known also as the effective viscosity) is a modified fluid viscosity which may be different from η. Frequently, $\eta' \approx \eta$. Both, η and η' are assumed to be constant. Equation (2) is completed by the potential U, which was absent in the original Brinkman's paper (BRINKMAN 1949).

Experimental measurements and computer simulation results have suggested that the Darcy-Brinkman equation should incorporate an effective viscosity.

Equation (2) can be written also in the form

$$- \nabla(p + U) = \frac{\eta}{K} v - \eta' \Delta v$$

more convenient for a discussion. We see that for low values of K, and small spatial variations of the velocity v this equation is approximated by Darcy's

Equation (1). For high values of K Stokes' equation for a steady flow of fluid with the viscosity η' is obtained (cf. LANDAU, LIFSHITZ 1987)

$$0 = -\nabla(p + U) + \eta' \Delta v$$

Equation (2) was proposed by Brinkman without giving any proof (BRINKMAN 1949).

Indirect, experimental proofs of validity of Brinkman's equation are provided by Gordon S. Beavers and Daniel D. Joseph (BEAVERS, JOSEPH 1967) and by Geoffrey Ingram Taylor (TAYLOR 1971). The proofs are more valuable as, apparently, these authors did not know Brinkman's paper (BRINKMAN 1949).

In theoretical way Brinkman's law was obtained by Enrique Sanchez--Palencia and Thérèse Lévy, who considered the fluid flow through an array of fixed particles (SANCHEZ-PALENCIA 1983, LÉVY 1983) and applied asymptotic expansions in series. These papers deal with idealized models of porous medium, represented by an array of rarely distributed balls. However, both Darcy's and Brinkman's laws are macroscopic ones, and some macroscopic argument seems to be needed in deriving Brinkman's equation.

Francisco J. Valdes-Parada et al. gave a theoretical back-up for the existence and meaning of an effective viscosity for the Stokes flow within a porous medium (VALDES-PARADA et al. 2007).These authors have shown that the use of a slip boundary condition is required to obtain an effective viscosity different from the one corresponding to the fluid phase. The proof is done by means of an up-scaling procedure based on volume averaging methods, which provides a boundary-value problem to compute the underlying effective viscosity (VALDES-PARADA et al. 2007, WHITAKER 1999).

The scattering process of the fluid flow against the porous canals walls suggests an idea of applying Born's approximation used favorably in description of scattering in quantum mechanics (BORN 1926, also MORSE, FESHBACH 1953, SHANKAR 1994). Born's approximation constitutes a version of successive approximation method. We apply it because Brinkman's equation is formally similar to Schrödinger's equation. This approximation, applied in quantum scattering theory consists of taking the incident field in place of the total field as the driving field at each point in the scatterer. In our case, the role of driving field is realised by Darcy's flow. The term *driving field* is explained in Appendix 1. Now, starting from Stokes' flow equation and Darcy's law, we are going to obtain Brinkman's law by Born's method as a correction of Darcy's law.

Brinkman's law derived by Born's approximation

Consider Stokes' equation with potential U

$$0 = -\nabla(p + U) + \eta\,\Delta v \tag{3}$$

and assume that the flow is incompressible, it is

$$\nabla \cdot v = 0 \tag{4}$$

We apply the operator ∇ to Equation (3), and by Equation (4) we get

$$\Delta\,(p + U) = 0 \tag{5}$$

The same harmonicity property has the sum $(p + U)$ in Darcy's law (1) and in Brinkman's law (2).

It was shown by asymptotic methods that Stokes' Equation (3) subject to periodic boundary conditions for the velocity v at walls of pores in a given porous medium leads to Darcy's law (1) (cf. SANCHEZ-PALENCIA 1980, WOJNAR 2014).

Hence, as a zero-th approximation of solution of Equation (3) for the flow in porous medium we take just Darcy's law

$$v^0 = -\frac{K^0}{\eta}\,\nabla(p + U) \tag{6}$$

where K^0 is a constant. According to the method of Born's approximation, we look for a corrected solution in the form

$$v = v^0 + \lambda\, v^1 \tag{7}$$

where λ is a small number, or, by (6)

$$v = -\frac{K^0}{\eta}\,\nabla(p + U) + \lambda\, v^1 \tag{8}$$

We substitute the relation (8) into Equation (3), and using harmonicity (5) of the sum $(p + U)$ we get

$$0 = -\nabla(p + U) + \eta\,\Delta\,(\lambda\, v^1) \tag{9}$$

This equation differs from Equation (3) only by notation of the velocity vector, here we have $\lambda\, v^1$ instead of v in (3). Thus, in analogy with the expression (6) the approximated solution of Equation (9) reads

$$\lambda\, v^1 = -\frac{K^1}{\eta}\nabla(p + U) \tag{10}$$

where K^1 is a new constant, or again by Equation (3)

$$\lambda\, v^1 = -\frac{K^1}{\eta}\,\eta\,\nabla v \tag{11}$$

Therefore, by (8) and (11) we find

$$v = -\frac{K^0}{\eta}\nabla(p + U) - \frac{K^1}{\eta}\,\eta\,\Delta\, v \tag{12}$$

Now, if we substitute $K^0 \equiv K$ and $K^1 = -Kv'/v$ we get Brinkman's Equation (2). Notice, that introducing the constant K^1 is equivalent to introducing the Brinkman's effective viscosity.

Non-homogeneous porous medium

Now, we show that proposed method of derivating Brinkman's equation can be applied to the linearly non-homogeneous porous medium, it is to the case

$$K = a_0 + a_1x_1 — a_2x_2 + a_3x_3 \tag{13}$$

where:
a_0 and a_i, $i = 1,2,3$ are constants,
while x_i, $i = 1,2,3$ are the position x components.

Just for clarity, apart from the direct symbolic vector notation, we will use the indicial notation. The subscripts range from 1 to 3, and Einstein's summation convention over repeated subscripts is observed.

For a non-homogeneous porous medium Darcy's law still reads

$$v_i^0 = -\frac{K}{\eta}\frac{\partial\,(p + U)}{\partial\, x_i} \tag{14}$$

but now the permeability K depends on the position $x = (x_1, x_2, x_3) = (x_i)$ with $i = 1,2,3$,

$$K = K(x)$$

and, naturally, the viscosity η is constant, it is x – independent. Hence, Stokes' Equation (3) holds

$$-\frac{\partial(p + U)}{\partial x_i,} + \eta \frac{\partial^2 v_i}{\partial x_k \partial x_k} = 0 \tag{15}$$

After applying ν – operator to both sides of the last equation, and using the incompressibility condition (4) we get, cf. Equation (5),

$$\frac{\partial^2 (p + U)}{\partial x_k \partial x_k} = 0 \tag{16}$$

On its turn, the incompressibility condition (4) expressed on (14) reads

$$\frac{\partial}{\partial x_i}\left(K \frac{\partial (p + U)}{\partial x_i}\right) = 0 \tag{17}$$

or

$$\frac{\partial K}{\partial x_i}\frac{\partial (p + U)}{\partial x_i} + K \frac{\partial^2 (p + U)}{\partial x_k \partial x_k} = 0$$

or, by (16)

$$\frac{\partial K}{\partial x_i}\frac{\partial (p + U)}{\partial x_i} = 0 \tag{18}$$

Hence, after differentiation

$$\frac{\partial K}{\partial x_i}\frac{\partial^2 (p + U)}{\partial x_k \partial x_i} = -\frac{\partial^2 K}{\partial x_k \partial x_i}\frac{\partial (p + U)}{\partial x_i} \tag{19}$$

As in the previous section, we look for a corrected Darcy's law in the form

$$v_i = -\frac{K^0}{\eta}\frac{\partial(p + U)}{\partial x_i} + \lambda\, v_i^1 \tag{20}$$

We submit the relation (20) into Stokes' Equation (3)

$$-\frac{\partial (p + U)}{\partial x_i} + \eta \frac{\partial^2}{\partial x_k\, \partial x_k}\left(-\frac{K^0}{\eta}\frac{\partial (p + U)}{\partial x_i} + \lambda\, {v^1}_i\right) = 0 \tag{21}$$

Now

$$\frac{\partial}{\partial x_k\, \partial x_k}\left(K^0 \frac{\partial(p + U)}{\partial x_i}\right) = \frac{\partial^2 K^0}{\partial x_k\, \partial x_k}\frac{\partial(p + U)}{\partial x_i} + 2\frac{\partial K^0}{\partial x_k}\frac{\partial^2(p + U)}{\partial x_k\, \partial x_i} + K^0 \frac{\partial^3(p + U)}{\partial x_k\, \partial x_k\, \partial x_i}$$

By Equation (16) the last term vanishes, and after substituting (19) we have

$$\frac{\partial^2}{\partial x_k\, \partial x_k}\left(K^0 \frac{\partial (p + U)}{\partial x_i}\right) = \frac{\partial^2 K^0}{\partial x_k\, \partial x_k}\frac{\partial(p + U)}{\partial x_i} - 2\frac{\partial^2 K^0}{\partial x_k\, \partial x_i}\frac{\partial(p + U)}{\partial x_k}$$

and by the linear relation (13) this whole term vanishes. Then, Equation (21) takes form

$$-\frac{\partial (p + U)}{\partial x_i} + \eta \frac{\partial^2(\lambda\, v_i^1)}{\partial x_k\, \partial x_k} = 0 \tag{22}$$

This is Stokes' type equation with unknown function $\lambda\, v_i^1$. Similarly as in the previous section, we regard Darcy's law to be an approximate solution of this equation

$$\lambda\, v_i^1 = -\frac{K^1}{\eta}\frac{\partial(p + U)}{\partial x_i} \tag{23}$$

or by Stokes' Equation (15)

$$\lambda\ v_i^1 = -\ \frac{K^1}{\eta}\ \eta\ \frac{\partial^2\ v_{\rm i}}{\partial x_k\ \partial x_k} \tag{24}$$

and after substitution $K^0 \equiv K$ and $K^1 = -K\ \eta'/\eta$ the corrected Darcy's Equation (20) reads

$$v_i\ =-\ \frac{K}{\eta}\left(\frac{\partial(p\ +\ U)}{\partial x_i}\ +\ \eta'\ \frac{\partial^2\ v_i}{\partial x_k\ \partial x_k}\right) \tag{25}$$

This is Brinkman's equation again.

Anisotropic viscous fluid

For such a fluid Darcy's law is of the form

$$v_i\ =-\ \kappa_{ij}\ \frac{\partial(p\ +\ U)}{\partial x_j} \tag{26}$$

where κ_{ij} is a symmetric matrix of constant coefficients, and $x = (x_i)$, $i = 1,2,3$ denotes the position. By incompressibility (4) we get

$$\kappa_{ij}\ \frac{\partial(p\ +\ U)}{\partial x_i\ \partial x_j}\ =\ 0 \tag{27}$$

For flow of anisotropic fluid, Stokes' equation is of the form (cf. LANDAU, LIFSHITZ 1987),

$$-\ \frac{\partial(p\ +\ U)}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\eta_{ijmn}\ \frac{\partial v_m}{\partial x_n}\right)\ =\ 0$$

where η_{ijmn} is the viscosity tensor. It satisfies the symmetry conditions

$$\eta_{ijmn}\ =\ \eta_{mnij}\ =\ \eta_{jimn}\ =\ \eta_{ijnm}$$

For constant coefficients η_{ijmn} we get the following form of Stokes' equation

$$-\frac{\partial(p + U)}{\partial x_i} + \eta_{ijmn}\frac{\partial^2 v_m}{\partial x_j\,\partial x_n} = 0 \tag{28}$$

We apply operator $\kappa_{ik}\,\partial/\partial\,x_k$ to both sides of the last equation, and get

$$-\kappa_{ik}\frac{\partial^2(p + U)}{\partial x_k\,\partial x_i} + \kappa_{ik}\,\eta_{ijmn}\frac{\partial^2 v_m}{\partial x_n\,\partial x_n\,\partial x_n} = 0 \tag{29}$$

or by (27)

$$\kappa_{ik}\,\eta_{ijmn}\frac{\partial^2 v_m}{\partial x_n\,\partial x_n\,\partial x_n} = 0$$

We look for a corrected seepage equation in the form

$$v_i = -\kappa_{ij}\frac{\partial(p + U)}{\partial x_j} + \lambda\,v_i^1 \tag{30}$$

where v^1 is a correction. We submit the expression (30) into Stokes' Equation (28) and receive

$$-\frac{\partial(p + U)}{\partial x_i} + \eta_{ijmn}\frac{\partial^2}{\partial x_j\,\partial x_n}\left(-\kappa_{mk} - \frac{\partial(p + U)}{\partial x_k} + \lambda\,v_m^1\right) = 0$$

or

$$-\frac{\partial(p + U)}{\partial x_i} + \eta_{ijmn}\,\kappa_{mk} - \frac{\partial^3(p + U)}{\partial x_j\,\partial x_n\,\partial x_k} + \eta_{ijmn}\frac{\partial^2(\lambda\,v_m^1)}{\partial x_j\,\partial x_n} = 0 \tag{31}$$

If the vector r defined as

$$r_i \equiv \eta_{ijmn}\,\kappa_{mk}\frac{\partial^3(p + U)}{\partial x_j\,\partial x_n\,\partial x_k} \tag{32}$$

vanishes, it is, if

$$r_i = 0 \tag{33}$$

Equation (31) takes the Stokesian form, cf. Equation (28),

$$-\frac{\partial(p + U)}{\partial x_i} + \eta_{ijmn}\frac{\partial^2(\lambda\, v_m^1)}{\partial x_i\,\partial x_n} = 0 \tag{34}$$

and we arrive at the situation, similar to that after Equation (9) for the isotropic problem. Thus, the velocity correction v^1 satisfies Darcy's type equation

$$\lambda\, v_i^1 = -\,\kappa_{ij}\frac{\partial(p + U)}{\partial x_j}$$

or, by (34)

$$\lambda\, v_i^1 = -\,\kappa_{ij}\;\eta_{kjmn}\frac{\partial^2 v_m}{\partial x_j\,\partial x_n} \tag{35}$$

Substitution into (30) gives

$$v_i = -\,\kappa_{ik}\left(\frac{\partial(p + U)}{\partial x_k} + \eta_{kjmn}\,\frac{\partial^2 v_m}{\partial x_j\,\partial x_n}\right) \tag{36}$$

what is a form of Brinkman's equation for seepage of an anisotropic fluid through a porous medium. But (36) was obtained with the assumption (33) only.

A discussion for cubic anisotropy of the fluid

For isotropic fluid

$$\eta_{ijmn} = \eta(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \left(\zeta + \frac{2}{3}\,\eta\right)\delta_{mn}\;\delta_{ij}$$

Thus, the number of non-zero moduli (the viscosities) for an isotropic fluid is two: η and ζ. We observe that now

$$\eta_{1111} - \eta_{1122} - 2\eta_{1212} = 0$$

The least number of non-zero moduli (the viscosity tensor components) for cubic anisotropy (the next most symmetric fluid after the isotropic one), obtained by a suitable choice of the co-ordinate axes is three (cf. LANDAU, LIFSHITZ 1970). We take the axes along the three fourth-order axes of symmetry. The symmetry is tetragonal, and there remain only three different moduli of viscosity: η_{1111}, η_{1122} and η_{1212}.

For the cubic symmetry the permeability tensor is isotropic (cf. LANDAU, LIFSHITZ 1970), it is $\kappa_{ij} = \kappa\, \delta_{ij}$ and the vector r defined by (32) is

$$r_i \equiv \kappa\, \eta_{ijkn} \frac{\partial^3 (p + U)}{\partial x_j\, \partial x_n\, \partial x_k}$$

For $i = 1$ and for the cubic symmetry of the tensor η_{ijmn} we get

$$r_1 = \kappa \frac{\partial}{\partial x_1} \left[\eta_{1111} \frac{\partial^2 (p + U)}{\partial x_1^2} + (2\eta_{1212} + \eta_{1122}) \left(\frac{\partial^2 (p + U)}{\partial x_2^2} + \frac{\partial^2 (p + U)}{\partial x_3^2} \right) \right] \quad (37)$$

and similar expressions for $i = 2$ and $i = 3$. Strictly speaking, only for an isotropic case, when

$$\eta_{1111} - 2\eta_{1212} - \eta_{1122}$$

the term in braces reduces to full Laplacian, but one may hope that components r_i, $i = 1,2,3$ can be neglected, when the difference $(\eta_{1111} - \eta_{1212} - 2\eta_{1122})$ is small. Thus, Brinkman's equation can be derived also for *nearly isotropic* fluids of cubic symmetry.

Conclusions

Applying an heuristic method of Born's approximation we have derived Brinkman's equation as a corrected Darcy's law. The derivation was given for isotropic Newtonian fluid for isotropic porous medium, with constant permeability, and with permeability linearly dependent on space coordinates. For anisotropic cubic fluids our methods works only in approximation.

Perhaps our method reconstructs Brinkman's reasoning. As it was mentioned in Introduction, the form of Brinkman's equation resembles somewise the time independent quantum wave equation, and Brinkman, who worked much in quantum physics surely knew Born's approximation method. Perhaps we have found the way of his arrival to his equation.

Appendix 1: Born's approximation

This method of successive approximation is a basic tool of calculus. It enables to solve a vast array of problems that other methods cannot handle. Also the scattering of particles on a potential $V(r)$ can be described by a successive approximation method, when we treat the potential as a perturbation. This is Born's approximation method (cf. also SAFRONO 2011, VALENTÍ 2014).

Time independent Schrödinger's wave equation can be written in the form

$$\Delta u + k^2 u - \lambda U(r)\, u = 0 \tag{38}$$

where

$$k^2 = \frac{2mE}{\mathrm{h}^2} \qquad \text{and} \qquad \lambda U = \frac{2mV}{\mathrm{h}^2}$$

Here k is a wave vector, $k = |k|$, and E is a total energy of the particle of the mass m. The parameter λ expresses the smallness of the disturbing term with the potential $V(r)$. The vector $r = (x, y, z)$ determines the position, and $r - |r|$. The value of the reduced Planck constant is: $\mathrm{h} = 1.054571800\ (13) \times 10^{-34}\ \mathrm{J \cdot s}$.

In zero-th approximation, it is for $\lambda = 0$ we have

$$u^0 = \mathrm{e}^{ikz}$$

as a solution of the equation

$$\Delta u^0 + k^2 u^0 = 0 \tag{39}$$

As the first approximation we take the sum

$$u = u^0 + \lambda u^1$$

We substitute this expression into (38) and after accounting for (39) we get

$$\Delta u_1 + k^2 u^1 = U(r)u^0 \tag{40}$$

This nonhomogeneous differential equation for the function u^1 has the general solution

$$u^1 = u^0 - \frac{1}{4\pi} \int \mathrm{d}\, r' \, \frac{\mathrm{e}^{ik}(r - r')}{| r - r' |} \, U(r') \, u^0(r')$$

The obtained result may be the starting point of the Born series.

In this example we see that in scattering theory and in particular in quantum mechanics, the Born approximation consists of taking the incident field in place of the total field as the *driving field* at each point in the scatterer.

Applications: The Born approximation is used in several different physical contexts.

In neutron scattering, the first-order Born approximation is almost always adequate, except for neutron optical phenomena like internal total reflection in a neutron guide, or grazing-incidence small-angle scattering.

The Born approximation has also been used to calculate conductivity in bilayer graphene (KOSHINO, ANDO 2006), and to approximate the propagation of long-wavelength waves in elastic media (GUBERNATIS et al. 1977).

Born's approximation conceived for scattering problems in quantum mechanics has been used extensively in seismological studies to describe seismic scattering by small-scale heterogeneities in the Earth. It is shown that geometrical ray effects, like the travel-time perturbation, ray bending and focusing, are contained within the Born scattering formalism, provided these effects are small (cf. HUDSON, HERITAGE 1981, COATES, CHAPMAN 1990).

Appendix 2: Meaning of Brinkman's effective viscosity

For the illustration of meaning of the effective viscosity η' in Brinkman's equation (2), let us consider steady flow of the liquid with a velocity v between two fixed parallel planes in the presence of a constant pressure gradient. It means that the pressure is a linear function of the coordinate x along the direction of flow. Let the exterior potential of volume forces vanish, $U = 0$. Hence

$$\frac{\partial p}{\partial x} \equiv - \gamma$$

where:
γ – constant.

We take one of these planes as the $x\ z$ – plane, with the x – axis in the direction of v. The distance between the planes is h, and the space for $-\infty < y < 0$ and $h < y < \infty$ is occupied by a porous medium with the permeability K and is permeated by the same liquid flowing under the same pressure gradient. It is clear that all quantities depend only on y, and that the fluid velocity is everywhere in the is x – direction. Thus, in the region $h > y > 0$ the flow is described by Stokes' equation

$$\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + \frac{\gamma}{\eta'} = 0 \tag{41}$$

The seepage in regions for $-\infty < y < 0$ and $h < y < \infty$ is described by Brinkman's equation

$$\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} - \frac{\eta}{\eta' K} v + \frac{\gamma}{\eta'} = 0 \tag{42}$$

The integration of Equations (41) and (42) gives:

in the region $-\infty < y < 0$

$$v = b_1\ \mathrm{e}^{a\ y} + \frac{K}{\eta} \gamma$$

in the region $0 < y < h$

$$v = -\frac{\gamma}{2\ \eta} y^2 + Cy + C_1$$

and in the region $h < y < \infty$

$$v = b_2\ \mathrm{e}^{-ay} + \frac{K}{\eta} \gamma,$$

where

$$a = \sqrt{\frac{\eta}{\eta' K}} \tag{43}$$

Both solutions in the regions $-\infty < y < 0$ and $h < y < \infty$ vanish for $y \to -\infty$ and for $y \to \infty$, respectively.

The constants b_1, b_2, C and C_1 are determined from the boundary conditions for $y = 0$ and $y = h$ in which the continuity of the velocity field and the shear strain rates is assumed. The result is

$$C = \frac{\gamma}{2\eta} h, \; C_1 = \frac{\gamma}{\eta}\left(\frac{h}{2a} + K\right), \quad b_1 = \frac{\gamma}{2\eta}\frac{h}{a} \quad \text{and} \quad b_2 = \frac{\gamma}{2\eta}\frac{h}{a}\,\mathrm{e}^{ah}.$$

Thus, the solution in the region $0 < y < h$ reads

$$v = \frac{\gamma}{2\eta}\left(h\,y - y^2 + \frac{h}{2a} + K\right)$$

and for the case of not permeable walls, it is for $K \to 0$, and in consequence $a \to \infty$, cf. the formula (43),

$$v \to \frac{\eta}{2\eta}(h - y)\,y$$

what represents Hagen-Poiseuille's type flow in two dimensions (cf. LANDAU, LIFSHITZ 1987).

According to the formula (43) the influence of the effective viscosity η' is similar. Vanishing of η' leads to the infinite value of the constant a, and the immediate extinction of the exponential terms in the solutions. Then in the regions $-\infty < y < 0$ and $h < y < \infty$ Darcy's seepage is existing only.

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